Some Subgroups of the Modular Group and Wicks Forms

Stacey Aston

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School of Mathematics & Statistics
Newcastle University
Newcastle upon Tyne
United Kingdom

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Abstract

The method for counting maximal orientable Wicks forms of given genus is discussed in detail, before exploring some of the history behind counting subgroups of the Modular group.

The main result is a one-to-one correspondence between some free subgroups of the Modular group $PSL(2, \mathbb{Z}) \cong C_2 \star C_3$ with index n and maximal orientable Wicks forms of length $n = 12g - 6$. This one-to-one correspondence has been achieved through the use of graph theoretic tools and combinatorics. The result allows one to give a closed formula for the number of conjugacy classes of a specific type of free subgroups in $PSL(2, \mathbb{Z})$ with given index.

Contents

Chapter 1

Introduction

1.1 Motivation

The Modular Group is an object that has been well studied in many areas of mathematics, having many implications in number theory and group theory. This infinite group is so complex that despite the wealth of knowledge already accumulated by mathematicians over the last two centuries, many of its features are still yet to be described in full. The feature that interests us are its subgroups; specifically, we look at a particular type (those with associated finite transitive triples) of free subgroups in the Modular Group and ask how many of these there are with given index. An answer to this question is desirable due to the current interest in subgroup growth, a branch of group theory that tackles the problem of counting subgroups.

1.2 Background

The Modular Group, or $PSL(2,\mathbb{Z})$ as it is often written, can be defined as the group of all linear fractional transformations of the upper half of the complex plane that have the form

$$z \mapsto \frac{az+b}{cz+d}, \text{ where } a,b,c,d \in \mathbb{Z} \text{ and } ad-bc=1,$$

where the group operation is function composition. It can be shown that G is generated by the transformations A and B where

$$Az = -\frac{1}{z},\ Bz = z+1.$$

Then, $A^2 = 1$ and $(AB)^3 = 1$. This allows one to show that $PSL(2,\mathbb{Z}) \cong C_2 * C_3$, where C_i is the cyclic group of order i, Serre (1973). Hence, for the duration of the thesis we consider the Modular Group as the group $M = C_2 * C_3$. A presentation for M is given by

$M = \langle m_1, m_2 | m_1^2 = m_2^3 = e \rangle$, where we use e to represent the identity element of M.

Study of the Modular Group began in the late 1800s. One of its first appearances of note was as part of the *Erlangen Program*, a research project of Felix Klein in 1872. The study of subgroups of the Modular Group became popular in the 1960s and was researched extensively by several group theorists until the 1980s when interest died down. Researchers (in particular Stothers and Millington) looked for a closed formula to count the number of subgroups of any given finite index. Stothers came closest to achieving this, producing a recurrence relation that counts the number of free subgroups with given index; we will work through this in Chapter 3.

One of the main tools used in the exploration of the Modular Group and its subgroups were *coset diagrams*.

Definition 1.1. Let G be a group with presentation $G = \langle X | R \rangle$. Take a subgroup H of index n in G and enumerate the cosets of H as $B_1, \ldots, B_n$. The *coset diagram* of H is a graph Γ such that:

(i) Γ has n vertices labelled $1, \ldots, n$, where the vertex labelled j corresponds to the coset B_j of H;

(ii) there is a directed edge labelled x_i, with $x_i \in X$, from the vertex labelled r to the vertex labelled s if and only if $x_i B_r = B_s$.

Coset Diagrams were originally defined for the Modular Group by Mushtaq (1983). They are finite quotients of the *Cayley graph* of a group.

Definition 1.2. Let G be a group with presentation $G = \langle X | R \rangle$. The *Cayley graph* of G is defined to be the graph $\Gamma = (G, E)$, where

$$E = \{(g, xg) | g \in G, x \in X\}.$$

In other words, the vertex set of the Cayley graph of G is exactly the elements of G and two vertices g and g' are connected by an edge if and only if there is an element of X such that $xg = g'$, or vice versa. Thus, considering the presentation of the Modular Group M, we see that its Cayley graph will be made up of edges labelled m_1 that form cycles of length dividing 2 and edges labelled m_2 that form cycles of length dividing 3. Hence, as coset diagrams of subgroups of M are quotients of this Cayley graph, we deduce that a coset diagram of any subgroup of M will also have this property. This feature of the coset diagrams of the Modular Group will be of great importance for proving the main result of the thesis. It will allow the use of tools from graph theory and combinatorics to investigate these structures and hence discover new information about the subgroups of the Modular Group that they relate to.

1.3 Method

We begin in Chapter 2 by introducing orientable Wicks forms. Orientable Wicks forms are words over an alphabet $E \cup E^{-1}$ that obey a set of rules. It is shown how orientable Wicks forms are in one-to-one correspondence with graphs embedded on compact orientable surfaces of genus g. This g is then also referred to as the genus of the Wicks form. Thinking of orientable Wicks forms in this way allows us to find upper and lower limits for the length of an orientable Wicks form of given genus. We find that the longest orientable Wicks forms of genus g will be of length $n = 12g - 6$ and we call these maximal orientable Wicks forms. Whilst deducing this upper limit we also show that maximal orientable Wicks forms are equivalent to cubic graphs (graphs in which every vertex has degree 3) embedded on compact orientable surfaces. The rest of the chapter utilises this fact to produce a formula for the number of maximal orientable Wicks forms of given genus - a result originally presented by Vdovina and Bacher in 2002. To do this we first acknowledge that the maximal orientable Wicks form from which a cubic graph is obtained can be used to endow each vertex of the graph with an orientation; either positive or negative. After showing that the maximal orientable Wicks form is represented as a cyclic Bi-Eulerian path (a closed path that traverses every edge in a graph exactly once in each direction, but never traverses an edge in the opposite direction immediately after the first pass of that edge) on the graph, we are able to say that a cubic graph with oriented vertices will only correspond to a maximal orientable Wicks form if one can find a cyclic Bi-Eulerian path on that graph. Thus, by looking at all possibilities for the local connectivity of the graph around both positive and negative vertices one can deduce what the structure of the graph must look like local to any vertex. This leaves us with three types of local connectivity to consider and it is easily justified that these three constructions must be the building blocks for all possible cubic graphs that correspond to maximal orientable Wicks forms. Thus, all cubic graphs corresponding to maximal orientable Wicks forms can be found by recursive attachment of these constructions. As some graphs can be found in multiple ways by a different combination of attachments, IH transformations are introduced to identify equivalent graphs. The chapter is then concluded by proving the main result of Vdovina and Bacher using a mixture of combinatorial and graph theoretic techniques to count all possible cubic graphs that can be formed.

The next chapter begins with a brief history of the Modular Group. Then, a result of Stothers is given, which provides the basis for the main result of the thesis. Stothers result uses coset diagrams of subgroups of the Modular Group to find a recurrence relation for the number of free subgroups of given index. To do this, we first define an analogue of coset diagrams - n-configurations. These can be either degenerate or non-degenerate. Placing an equivalence relation on all n-configurations leads to a one-to-one correspondence between them and subgroups of index n in the Modular Group. Non-degenerate n-configurations represent coset diagrams that correspond to free subgroups of the Modular Group of index n. Thus, as these non-degenerate n-configurations are defined in a specific way, we can use combinatorial techniques to count the number of non-degenerate n-configurations for each n and produce a recurrence relation for the number of free subgroups in the Modular Group of index n.

In the final chapter, we use both Vdovina and Bacher's result and Stothers' result to find a closed formula for the number of conjugacy classes of free subgroups in the Modular Group of index n with associated finite transitive triple. The first step is to place a second equivalence relation on the n-configurations. The new set of equivalence classes is then in one-to-one correspondence with conjugacy classes of subgroups with index n in the Modular Group. Thus, the non-degenerate n-configurations correspond to conjugacy classes of free subgroups. Some free subgroups of the Modular Group have an associated finite transitive triple. We say a free subgroup has a finite transitive triple if one can read off permutations A and B from its coset diagram (found by inspection of the action of each generator on the subgroups - represented by the edges of the coset diagram) such that $C = AB$ is transitive. The free subgroups of index n that have this property will hence be represented by the non-degenerate n-configurations that give rise to such a C. We will define a set of rules that allows one to pass from any such non-degenerate n-configuration to a cubic graph with cyclic orientation at the vertices. This set of rules amounts to shrinking 3-cycles and identifying the edges in 2-cycles to leave a cubic graph. One can then use the orientation of the 3-cycles to place an orientation on the vertices of the cubic graph obtained. As these rules can be defined in an algorithmic way, then it is possible to run the algorithm backwards to pass freely between non-degenerate n-configurations with associated finite transitive triple and cubic graphs with cyclic orientation at the vertices. The final step is to show that any cubic graph defined in such a way will have a cyclic Bi-Eulerian path, which can be found from the permutations A and B. From Chapter 2 we already know that cyclic Bi-Eulerian paths are equivalent to maximal orientable Wicks forms. Thus, we can count all such graphs and hence we can count the number of conjugacy classes of free subgroups of index n in the Modular Group that have an associated finite transitive triple, giving the main result.

Theorem 4.3 *There is a one-to-one correspondence between conjugacy classes of free subgroups of index* $n = 12g - 6$ *in* M*, with related finite transitive triple* (Ω, A, B)*, and MOWFs of genus* g*.*

Chapter 2

Orientable Wicks Forms

Consider an *alphabet* $A = \{a_1^{\pm 1}, a_2^{\pm 1}, \ldots\}$. A *word* over A is an expression of the form $w = w_1 w_2 \ldots w_{2l}$, where each $w_i = a_j^{\epsilon}$, for $\epsilon = \pm 1$. A *subword* of w is a word $u = w_i w_{i+1} \ldots w_{i+j}$, where $i + j \leq 2l$ and u appears in w. We will consider *cyclic words* $[w]$, defined as the set of all cyclic permutations of w. The following terminology will be required.

(i) The *length* of $[w]$ is $2l$.

(ii) The word u is a *factor* of $[w]$ if u is a subword of some element of $[w]$.

(iii) We say $[w]$ is *reduced* if $[w]$ has no factor of the form aa^{-1} or $a^{-1}a$.

Definition 2.1. An *orientable Wicks form* is a cyclic word $[w]$ over A such that:

(i) if $a^{\epsilon} \in A$ appears in w, where $\epsilon = \pm 1$, then $a^{-\epsilon}$ appears exactly once in w;

(ii) $[w]$ is cyclically reduced;

(iii) if $a_i^{\epsilon} a_j^{\delta}$ is a factor of $[w]$ then $a_j^{-\delta} a_i^{-\epsilon}$ is not a factor of $[w]$.

As we will always be dealing with cyclic words we abuse notation and write w in place of $[w]$ for the rest of the text.

2.1 Maximal Orientable Wicks Forms

When an orientable Wicks form w is thought of as an element of the free group $\mathbb{F}$, generated by the alphabet A, it is possible to show that w will be an element of the commutator subgroup. With this in mind, we may define the *algebraic genus* $g_a(w)$ of w as the smallest integer g_a such that w can be written as the product of g_a commutators in $\mathbb{F}$.

Example 2.1. Let w be an orientable Wicks form over the alphabet $\{a, b, c\}$ defined by $w = ac^{-1}ba^{-1}cb^{-1}$. It is an easy task to show that w can be written as a single commutator with respect to the given alphabet. Set $x = ac^{-1}$ and $y = ba^{-1}$. The commutator $xyx^{-1}y^{-1} = ac^{-1}ba^{-1}ca^{-1}ab^{-1} = ac^{-1}ba^{-1}cb^{-1}$. Thus, $g_a(w) = 1$.

An orientable Wicks form w can also be thought of as a graph Γ embedded on an orientable compact surface S. The surface of least genus that w can be embedded on can be found by labelling and orienting the boundary of a disc according to w. We then identify the arcs of the disc (respecting orientation) to form an orientable compact surface S with embedded graph Γ. The genus of this surface is defined as the *topological genus* $g_t(w)$ of w.

Example 2.2. Again, we consider the orientable Wicks form $w = ac^{-1}ba^{-1}cb^{-1}$ over the alphabet $\{a, b, c\}$. Fig. 1 shows how we can write w on the boundary of a disc in order to form torus with a twist; which is a surface of genus 1.

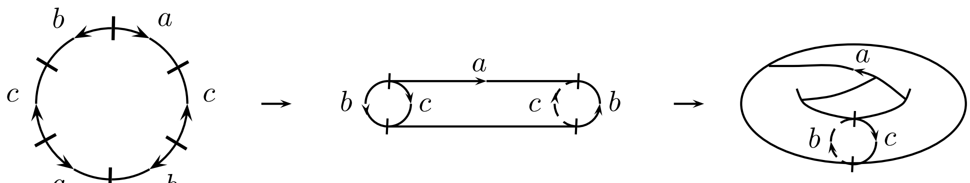


Fig. 1

Thus, $g_t(w) = 1$.

Notice that in the previous two examples we found that $g_a(w) = g_t(w)$ for the orientable Wicks form $w = ac^{-1}ba^{-1}cb^{-1}$. It can be shown that this is true in general.

Proposition 2.1. *The algebraic genus and topological genus of an orientable Wicks form coincide, Culler (1981); Comerford & Edmunds (1994).*

As $g_a(w)$ and $g_t(w)$ are always equal we will only speak of the genus g of w from here on, where it is understood that one can consider g to be both the algebraic and topological genus of w.

Thinking of an orientable Wicks form w as a graph Γ embedded on a surface S allows us to deduce upper and lower limits for the length $2l$ of w that has given genus g. To do this, we first observe that if w has length $2l$, then Γ will have l edges, by noting that Γ is obtained by identifying the $2l$ arcs that make up the boundary of a disc that has been labelled and oriented according to w (we note also that $S\backslash\Gamma$ is necessarily connected and simply connected). Condition (i) from the definition of an orientable Wicks form ensures that each label appears exactly twice on the boundary of the disc, having opposite orientation in each case. Thus, when all arcs have been identified, we are left with l distinct edges. We can infer further properties of Γ by contemplating the other two conditions in Definition 2.1. Consider the two types of identification in Fig. 2. Condition (ii) guarantees that Γ has no vertices of degree 1, as there can be no identifications of type 1. Whereas condition (iii) ensures that there will also be no vertices of degree 2, as no identifications of type 2 will occur.

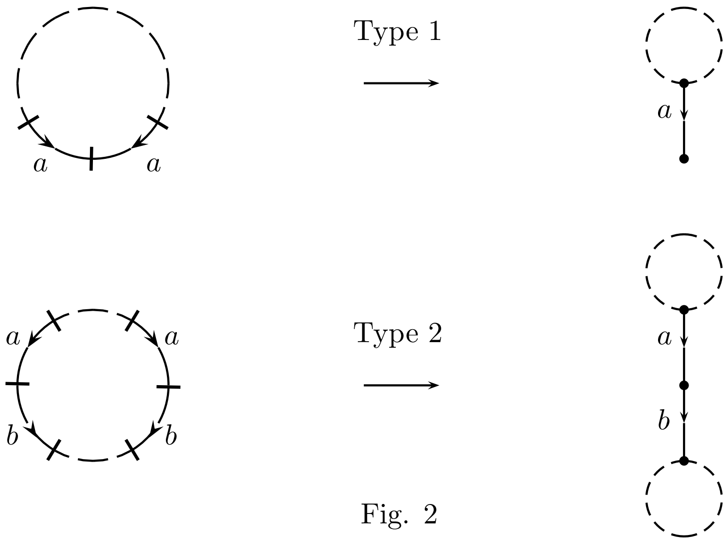


Fig. 2

In fact, orientable Wicks forms are in one-to-one correspondence with graphs embedded on orientable compact surfaces, having no vertices of degree 1 or 2. We have already seen how any orientable Wicks form can be used to find an orientable compact surface S, with embedded graph Γ that has no vertices of degree 1 or 2 (where $S \backslash \Gamma$ is connected and simply connected). Let us now suppose that we have such a surface S with embedded graph Γ, that has no vertices of degree 1 or 2. We show that it is possible to obtain an orientable Wicks form from such an S.

Cutting S along Γ produces a disc with a word w written on its boundary. Let E be the edge set of Γ, then w will be a word over the alphabet $E \cup E^{-1}$. Careful consideration allows one to conclude that:

(i) by cutting along every edge we ensure w contains each e_i in E exactly once and each e_i^{-1} in E^{-1} exactly once;

(ii) as Γ has no vertices of degree 1, w will contain no subword of the form $e_i e_i^{-1}$ or $e_i^{-1} e_i$;

(iii) as Γ has no vertices of degree 2, w will only have a subword of the form $e_i^{\epsilon} e_j^{\delta}$ if w does not contain a subword of the form $e_i^{-\delta} e_i^{-\epsilon}$.

Properties (i), (ii) and (iii) above of w match conditions (i), (ii) and (iii) from Definition 2.1, respectively. Thus, w is an orientable Wicks form.

Previously, we noted that if S is the surface obtained from the orientable Wicks form $w = w_1 w_2 \ldots w_{2l}$, then Γ must have l edges. Let n represent the number of vertices of Γ and V represent the set of all vertices of Γ. Using the Euler characteristic of S we can find a formula for the number of edges l of Γ in terms of n and the genus g of S. The Euler characteristic of S, $\chi(S)$, can be defined in two ways. We can define $\chi(S)$ in terms of the genus of S as

$$\chi(S) = 2 - 2g$$

or, we can define $\chi(S)$ in terms of the embedded graph Γ as

$$\chi(S) = n - l + 1.$$

Equating the above two equalities and rearranging for l gives

$$l = 2g + n - 1.$$

This equation can now be used to find upper and lower limits for l that depend only on g. Or in other words, we can give a formula for the maximum and minimum lengths of an orientable Wicks form w of given genus g. First, we find the lower limit for the number of edges in Γ. As Γ is connected, adding extra vertices to Γ means adding extra edges. Thus, the minimum number of edges will occur when we have only one vertex. To find this minimum number of edges we can set n equal to 1 in our equation for l. We obtain

$$l = 2g.$$

Let us now consider the upper limit. The maximum number of edges will occur in a graph where all vertices have degree 3. To see this, suppose we have a graph Γ embedded on S with genus g such that some vertex v of Γ has degree greater than 3. Then, around v the structure of Γ will be as seen in Fig 3, which shows that we can add an extra edge to Γ to create two vertices v_1 and v_2 that both have degree less than the degree of v.

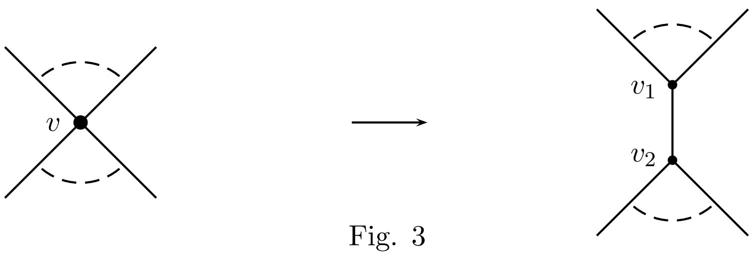


Fig. 3

We can repeat this process so that we have an embedded graph Γ on S whose vertices have as low a degree as we would like. However, we already know that an embedded graph Γ has no vertices of degree 1 or 2 so, the maximum number of edges must occur when we have a graph Γ whose vertices all have degree 3.

The *Handshaking Lemma* states that

$$\sum_{v \in V} \deg(v) = 2l.$$

We wish to maximise l and hence $2l$ so we suppose that $\deg(v) = 3$ for all v in V. Thus, the Handshaking Lemma becomes

$$3n = 2l \Rightarrow n = \frac{2l}{3}.$$

Substituting this into our formula for l gives

$$l = 6g - 3.$$

As we have found upper and lower bounds for l, we have also found upper and lower bounds for the length $2l$ of an orientable Wicks form of given genus g. We have that

$$4g \leq 2l \leq 12g - 6.$$

We will say an orientable Wicks form w is *maximal* if it has length $2l = 12g - 6$. We refer to such a w as a *maximal orientable Wicks forms (MOWF).*

Remark 2.1. We note that the graph Γ related to a MOWF w must have $6g - 3$ edges and $(2(6g - 3))/3 = 4g - 2$ vertices of degree 3.

2.2 Counting Maximal Orientable Wicks forms

We continue to consider an orientable Wicks form as a graph Γ embedded on a compact connected surface S. If Γ is a graph corresponding to a MOWF, then the vertices of Γ can be assigned with an orientation according to the following rules:

(i) a vertex v of Γ, that has directed edges e_1, e_2 and e_3 leaving v, will be called *positive* if $w = e_1^{-1} e_2 \ldots e_2^{-1} e_3 \ldots e_3^{-1} e_1$ or $w = e_1^{-1} e_3 \ldots e_3^{-1} e_2 \ldots e_2^{-1} e_1$;

(ii) a vertex v of Γ, that has directed edges e_1, e_2 and e_3 leaving v, will be called *negative* if $w = e_1^{-1} e_2 \ldots e_3^{-1} e_1 \ldots e_2^{-1} e_3$ or $w = e_1^{-1} e_3 \ldots e_2^{-1} e_1 \ldots e_3^{-1} e_2$.

In order to count MOWFs we will investigate their associated graphs. We have already seen that a MOWF will come from a cubic graph Γ (where the word *cubic* implies all vertices of Γ have degree 3). We will consider what implications the orientation (positive or negative) of some vertex v of Γ has on the structure of Γ around v by exhausting all possibilities for the connectivity of the vertices neighbouring v. This structure will impact on the behaviour of the associated MOWF (if it is possible to find one). In the case that there is an associated MOWF, it will be found by constructing a *cyclic Bi-Eulerian path* on the associated graph Γ.

Definition 2.2. A *Bi-Eulerian path* on a graph is defined as a closed path that traverses every edge and its inverse exactly once, without traversing any edge followed immediately by its inverse. A *cyclic Bi-Eulerian path* on a graph is a Bi-Eulerian path where every cyclic permutation of the path is also a Bi-Eulerian path.

Lemma 2.1. *There is a one-to-one correspondence between cyclic Bi-Eulerian paths on cubic graphs $\Gamma = (V, E)$ and MOWFs over the alphabet $E \cup E^{-1}$.*

Proof. See proof of Lemma 4.1. ☐

Lemma 2.1 facilitates a second definition of cyclic Bi-Eulerian paths.

Definition 2.3. A *cyclic Bi-Eulerian path* is a path p in a graph $\Gamma = (V, E)$ such that:

(i) e_i and e_i^{-1} appear exactly once in p for all $e_i \in E$;

(ii) p is cyclically reduced;

(iii) if $e_i^{\epsilon} e_j^{\delta}$ appears in p then $e_j^{-\delta} e_i^{-\epsilon}$ does not.

Remark 2.2. We note that in order to show that a cyclic Bi-Eulerian path satisfies (iii) from Definition 2.1, we actually proved that if a cubic graph Γ has a cyclic Bi-Eulerian path then there can be no loops in Γ. This is why we do not consider loops as possible components of the graphs that follow immediately in this section.

Let v be a positive vertex of a cubic graph Γ.

Case P_1: v has only one neighbour.

In this case, we have Fig. 5.

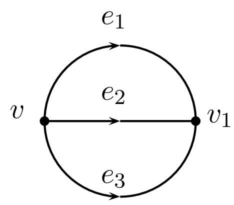


Fig. 5

All vertices have degree 3 already. Therefore, we cannot add any extra edges to Γ without causing Γ to be disconnected. This means we are looking at the whole of the graph in this case. As v is a positive vertex there cannot be a cyclic Bi-Eulerian path in Γ; as any path must be of the form

$$\begin{aligned} p &= e_1^{-1} e_2 e_2^{-1} e_3 e_3^{-1} e_1 \text{ or,} \\ p &= e_1^{-1} e_3 e_3^{-1} e_2 e_2^{-1} e_1. \end{aligned}$$

In both of these options for p we traverse an edge followed directly by its inverse. This is a contradiction to (ii) in Definition 2.3.

Case P_2: v has two neighbours.

(i) *The two neighbours are not adjacent.*
Here, we have Fig. 6.

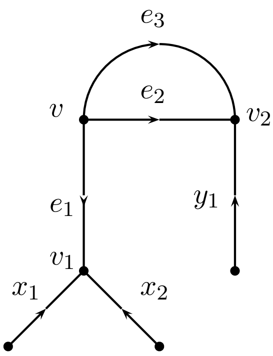


Fig. 6

Edges x_1, x_2 and y_1 have been added in to ensure all vertices adjacent to v have degree 3. Obeying the orientation at v we may form the paths

$$p = x_1e_1^{-1}e_2y_1^{-1}u_1y_1e_2^{-1}e_3e_3^{-1}e_1x_2^{-1}u_2x_2x_1^{-1}u_3 \text{ or,}$$
$$p = x_1e_1^{-1}e_3y_1^{-1}u_1y_1e_3^{-1}e_2e_2^{-1}e_1x_2^{-1}u_2x_2x_1^{-1}u_3.$$

These two paths are the closest we can get to obtaining a cyclic Bi-Eulerian path. However, we cannot satisfy condition (iii) of Definition 2.3, as both $e_2y_1^{-1}$ and $y_1e_2^{-1}$ appear in p.

(ii) *The two neighbours are adjacent.*
This case presents us Fig 7.

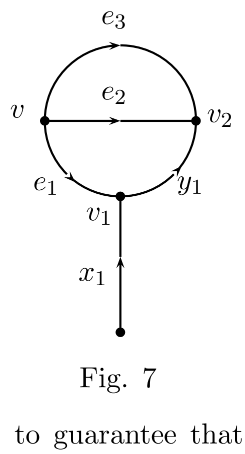


Fig. 7

Again, we add edges x_1 and y_1 to guarantee that all adjacent vertices to v have degree 3. The paths

$$p = e_1^{-1}e_2y_1^{-1}x_1^{-1}u_1x_1y_1e_2^{-1}e_3e_3^{-1}e_1 \text{ or,}$$
$$p = e_1^{-1}e_3y_1^{-1}x_1^{-1}u_1x_1y_1e_3^{-1}e_2e_2^{-1}e_1$$

are the closet we can get to a cyclic Bi-Eulerian path here. It is not possible to satisfy (ii) in Definition 2.3.

Case P_3: v has three neighbours.

(i) *None of the three neighbours are adjacent.*
We have Fig. 8.

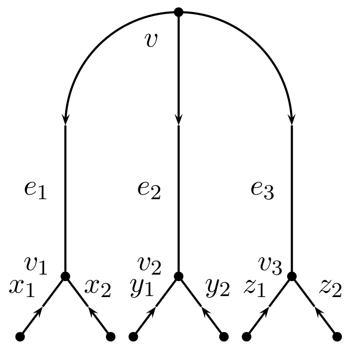


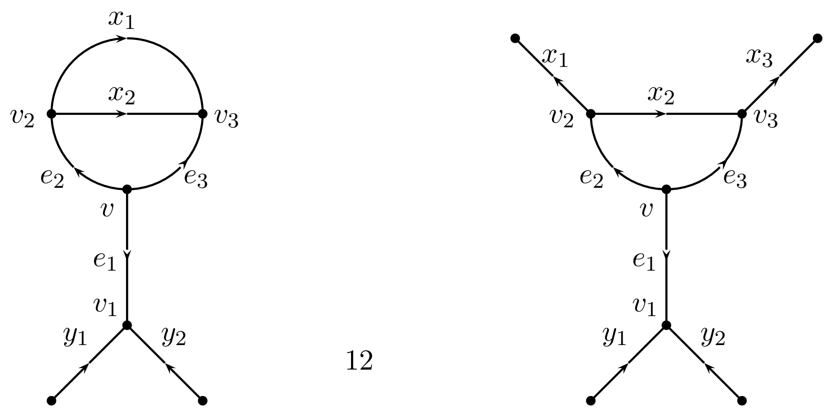
Fig. 8

We have added six extra edges for the same reasons as above. Obeying the rules set out by the positive vertex v, possible paths are

$$p = x_1e_1^{-1}e_2y_1^{-1}u_1y_1y_2^{-1}u_2y_2e_2^{-1}e_3z_1^{-1}u_3z_1z_2^{-1}u_4z_2e_3^{-1}e_1x_2^{-1}u_5x_2x_1^{-1}u_6 \text{ or,}$$
$$p = x_1e_1^{-1}e_3z_1^{-1}u_1z_1z_2^{-1}u_2z_2e_3^{-1}e_2y_1^{-1}u_1y_1y_2^{-1}u_2y_2e_2^{-1}e_1x_2^{-1}u_5x_2x_1^{-1}u_6.$$

Both of these paths satisfy Definition 2.3. Thus, in this case, we can find a cyclic Bi-Eulerian path. We note for later that v_1, v_2 and v_3 are also all positive vertices (other cyclic Bi-Eulerian paths can also be found in this case that result in a selection of v_1, v_2 and v_3 being negative).

(ii) *Only two of the three neighbours are adjacent.*
This case gives two options, seen in Fig. 9.



Option 1 Option 2

Fig. 9

Option 1 gives the routes

$$\begin{aligned} p &= y_1e_1^{-1}e_2x_1x_2^{-1}e_2^{-1}e_3x_1^{-1}x_2e_3^{-1}e_1y_2^{-1}u_1y_2y_1^{-1}u_2 \text{ or,} \\ p &= y_1e_1^{-1}e_3x_1^{-1}x_2e_3^{-1}e_2x_1x_2^{-1}e_2^{-1}e_1y_2^{-1}u_1y_2y_1^{-1}u_2. \end{aligned}$$

Whereas option 2 gives us

$$\begin{aligned} p &= y_1e_1^{-1}e_2x_2x_3u_1x_1^{-1}e_2^{-1}e_3x_2^{-1}x_1u_2x_3^{-1}e_3^{-1}e_1y_2^{-1}u_3y_2y_1^{-1}u_4 \text{ or,} \\ p &= y_1e_1^{-1}e_3x_2^{-1}x_1u_1x_3^{-1}e_3^{-1}e_2x_2x_3u_2x_1^{-1}e_2^{-1}e_1y_2^{-1}u_3y_2y_1^{-1}u_4. \end{aligned}$$

All of these paths are cyclic Bi-Eulerian paths. Note that the paths for Option 1 and Option 2 imply v_2 and v_3 are negative vertices.

(iii) *Each of the three neighbours is adjacent to the other two.*
The configuration around v is as seen in Fig. 10.

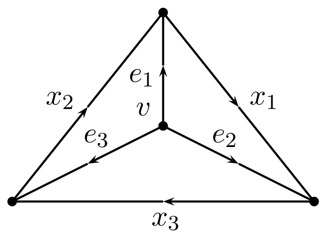


Fig. 10

It is clear here that there is no way to traverse each edge and its inverse exactly once without following an edge immediately by its inverse. Thus, we are unable to find a Bi-Eulerian path.

If we instead let v be a negative vertex of a cubic graph Γ, we can follow the same process as for a positive vertex to infer how Γ will be structured around v. In each case, the picture will be the same as before. Hence, as we already have these pictures on the previous pages, we will simply state the possible forms of w in each situation.

Case N_1: v has only one neighbour.
This scenario gives

$$\begin{aligned} p &= e_1^{-1}e_2e_3^{-1}e_1e_2^{-1}e_3\text{or} \\ p &= e_1^{-1}e_3e_2^{-1}e_1e_3^{-1}e_2. \end{aligned}$$

Both of these satisfy the conditions set out in Definition 2.3. Note that the paths imply v_1 is also negative.

Remark 2.3. This case actually yields a full cubic graph Γ that relates to a MOWF of genus 1 (as $p = w$ has length $6 = 12(1) - 6$.). Notice that this MOWF is identical to the orientable Wicks form we used in Examples 2.1 and 2.2 if we relabel the edges of Γ. We will see that this is in fact the unique MOWF of genus 1.

Case N_2: v has two neighbours.

(i) *The two neighbours are not adjacent.*
We have the following;

$$
\begin{aligned}
p &= x_1e_1^{-1}e_2e_3^{-1}e_1x_2^{-1}u_1y_1e_2^{-1}e_3y_1^{-1}u_2 \text{or} \\
p &= x_1e_1^{-1}e_3e_2^{-1}e_1x_2^{-1}u_1y_1e_3^{-1}e_2y_1^{-1}u_2.
\end{aligned}
$$

Assuming x_1^{-1} and x_2 appear as part of either u_1 or u_2, we have found cyclic Bi-Eulerian paths. We note that v_2 is negative, whereas v_1 and the initial vertex of y_1 are positive.

(ii) *The two neighbours are adjacent.*
Here, we obtain the paths

$$
\begin{aligned}
p &= x_1e_1^{-1}e_2e_3^{-1}e_1y_1e_2^{-1}e_3y_1^{-1}x_1^{-1}u_1 \text{or} \\
p &= x_1e_1^{-1}e_3e_2^{-1}e_1y_1e_3^{-1}e_2y_1^{-1}x_1^{-1}u_1.
\end{aligned}
$$

Again, we have found cyclic Bi-Eulerian paths. Note that v_2 is negative and v_1 and the initial vertex of x_1 are positive.

Case N_3: v has three neighbours.

(i) *None of the three neighbours are adjacent.*
This case gives the subsequent routes in Γ.

$$
\begin{aligned}
p &= x_1e_1^{-1}e_2y_2^{-1}u_1z_1e_3^{-1}e_1x_2^{-1}u_2y_1e_2^{-1}e_3z_2^{-1}u_3 \text{or} \\
p &= x_1e_1^{-1}e_3z_2^{-1}u_1y_1e_2^{-1}e_1x_2^{-1}u_2z_1e_3^{-1}e_2y_2^{-1}u_3.
\end{aligned}
$$

If we allow for the possibility that x_1^{-1}, x_2, y_1^{-1}, y_2, z_1^{-1} and z_2 appear as part of either u_1, u_2 and u_3 then we have cyclic Bi-Eulerian paths.

(ii) *Only two of the three neighbours are adjacent.*
Inspection of the associated figure for Option 1 quickly brings us to the conclusion that it is not possible to find a Bi-Eulerian path when v in negative.

Option 2 yields

$$
\begin{aligned}
p &= y_1e_1^{-1}e_2x_2e_3^{-1}e_1y_2^{-1}u_1x_1^{-1}e_2^{-1}e_3x_3u_2x_3^{-1}x_2^{-1}x_1u_3 \text{or} \\
p &= y_1e_1^{-1}e_3x_2^{-1}e_2^{-1}e_1y_2^{-1}u_1x_3^{-1}e_3^{-1}e_2x_1u_2x_1^{-1}x_2x_3u_3.
\end{aligned}
$$

These paths suffice as cyclic Bi-Eulerian paths if we suppose y_1 and y_2^{-1} appear as part of either u_1, u_2 or u_3. Note that the paths imply that the terminal vertices of x_1, x_3, y_1 and y_2 must all be positive with three distinct neighbours.

(iii) *Each of the three neighbours is adjacent to the other two.*
As in case P_3 (iii), it is impossible to construct a cyclic Bi-Eulerian path given these conditions on a cubic graph.

To summarize, given a cubic graph Γ, we are left with the following possibilities for its structure local to a given vertex v. Either v is positive and we are in case P_3 (i), or v is negative and we are in one of the cases N_1, N_2 (i), N_2 (ii), N_3 (i) or N_3 (ii). We need not consider option 1 from case P_3 (ii) as this is analogous to case N_2 (ii). We also need not consider option 2 from case P_3 (ii) as this is the same as option 2 in case N_3 (ii).

Remember that we are interested in cubic graphs embedded on surfaces. If we choose to only consider graphs sitting on surfaces of genus $g > 1$ we may ignore case N_1 (the unique graph when $g = 1$).

In case P_3 (i), for the cyclic Bi-Eulerian paths stated, we observed that the three distinct vertices adjacent to v must all also be positive. As these three vertices v_1, v_2 and v_3 are included in Γ too, they must have three distinct neighbouring vertices, all of which are positive. This argument could be continued to find an endless string of positive vertices. Thus, if we are in case P_3 (i), with a path of this form, Γ must be infinite. Therefore, we may neglect this case and assume that we are required to choose a p that will assign negative orientation to either v_1, v_2 or v_3.

For case N_3 (i), we are forced to build the graph outwards to vertices that all are positive with three distinct neighbours. By the above argument, a positive vertex must have a negative neighbour. If this negative neighbour implies we are back in this case then we have a repeat of the situation just described. Hence, the graph would continue to grow until we find ourselves in either case N_2 (i), case N_2 (ii) or case N_3 (ii). Thus, we can ignore this case and assume it occurs as one of the other options.

We have deduced that we only need to consider the structure of Γ local to a negative vertex v; as we have shown that any cubic graph that has a Bi-Eulerian path must possess a negative vertex that satisfies the conditions of one of the remaining three cases. Simplifying these cases, we have that this structure must be one of three kinds. Let us say that these different structures come from three different types of negative vertices. Call these *type* α, *type* β and *type* γ (see Fig. 11).

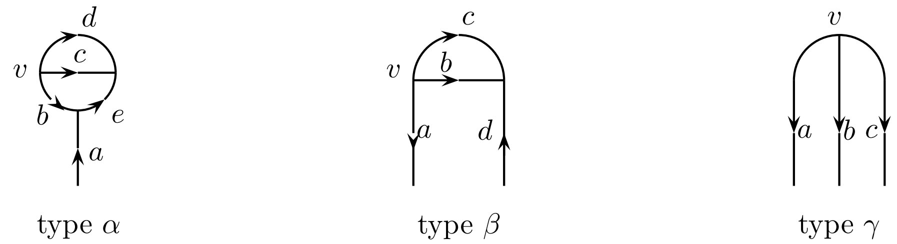


Fig. 11

We conclude that all cubic graphs ommitting Bi-Eulerian paths can be found by attaching some combination of type α, β or γ vertices to the graph observed in case N_1; which we will now refer to as the *genus* 1 *graph*. Relabelling the possible configurations of Γ around v, as we have, allows for simplification of the MOWFs that one would find in each case.

Type α**:** The type α vertex is representative of Case N_2 (ii), where the vertex v has only two neighbours and these neighbours are adjacent to each other. We have that any MOWF obtained from a graph Γ, that has a type α vertex v, must have the form

$$w = x_1ab^{-1}cd^{-1}bec^{-1}de^{-1}a^{-1}x_2^{-1}u_1x_2x_1^{-1}u_2.$$

Note that we have assumed edges x_1 and x_2 are the other two edges that are entering the initial vertex of a. The addition of this structure to a cubic graph that already emits a MOWF will be referred to as an *α-construction*. Assume the resulting MOWF w (given above) has genus g. This kind of construction implies w can be obtained from the MOWF w' of genus $g-1$, where

$$w' = xu_1x^{-1}u_2,$$

by the substitutions

$$\begin{aligned} x &\mapsto x_1ab^{-1}cd^{-1}bec^{-1}de^{-1}a^{-1}x_2^{-1}, \\ x^{-1} &\mapsto x_2x_1^{-1}. \end{aligned}$$

Suppose the cubic graph relating to w' is Γ' and the graph relating to w is Γ. The construction is seen in Fig. 12.

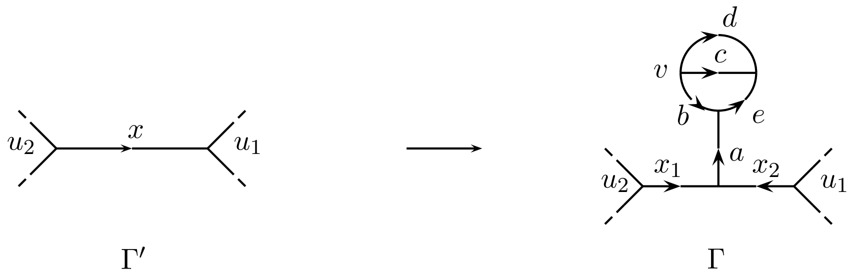


Fig. 12

Type β**:** This type of vertex comes from case N_2 (i). The case where v has only two neighbours that are not adjacent to each other. A MOWF related to a cubic graph that has a type β vertex will take the form

$$w = x_1 a^{-1} b c^{-1} a x_2^{-1} u_1 y_1 d b^{-1} c d^{-1} y_2^{-1} u_2.$$

Here, we suppose x_1 and x_2 are the edges entering the terminal vertex of a and y_1 and y_2 are edges entering the initial vertex of d. We also assume x_1^{-1}, x_2, y_1^{-1} and y_2 appear as part of u_1 or u_2. We recognise that it is possible to have $x_2^{-1} = y_1$ and/or $x_1 = y_2^{-1}$. Assuming again that w has genus g, we say that w is obtained from the genus $g-1$ MOWF

$$w' = x u_1 y u_2$$

by the substitutions

$$\begin{aligned} x &\mapsto x_1 a^{-1} b c^{-1} a x_2^{-1}, \\ y &\mapsto y_1 d b^{-1} c d^{-1} y_2^{-1}. \end{aligned}$$

This will be referred to as a β-*construction*. Once again we let w' arise from Γ' and w from Γ in order to illustrate this graphically in Fig. 13.

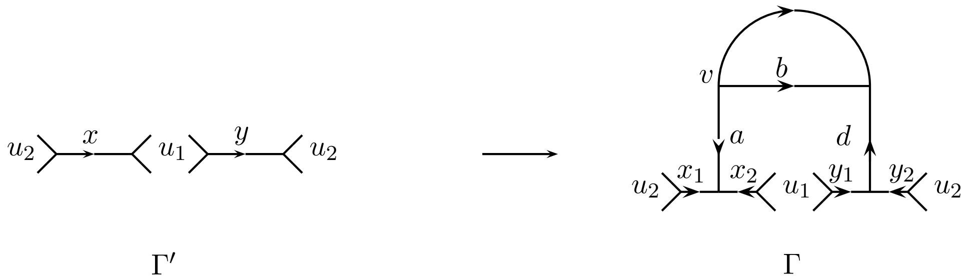


Fig. 13

Type γ**:** This final type represents case N_3 (i), where v has three vertices, all of which are adjacent. Here, a MOWF would have the form

$$w = x_1 a^{-1} b y_2^{-1} u_1 z_1 c^{-1} a x_2^{-1} u_2 y_1 b^{-1} c z_2^{-1} u_3,$$

where it is presumed x_1 and x_2 are the two other edges entering the terminal vertex of a; y_1 and y_2 are the two other edges entering the terminal vertex of b; z_1 and z_2 are the two other edges entering the terminal vertex of c. Notice that some identifications may be possible here too. We may adopt any combination of the following: $x_1 = z_2$, $y_1 = x_2$, $z_1 = y_2$. If this type of structure is present we say w is obtained from

$$w' = x u_2 y u_1 z u_3$$

(again this is a MOWF of genus one less than w) by a γ*-construction.* This type of construction amounts to performing the substitutions

$$\begin{aligned} x &\mapsto x_1 a^{-1} b y_2^{-1}, \\ y &\mapsto z_1 c^{-1} a x_2^{-1}, \\ z &\mapsto y_1 b^{-1} c z_2^{-1}. \end{aligned}$$

Finally, we illustrate this scenario in Fig. 14.

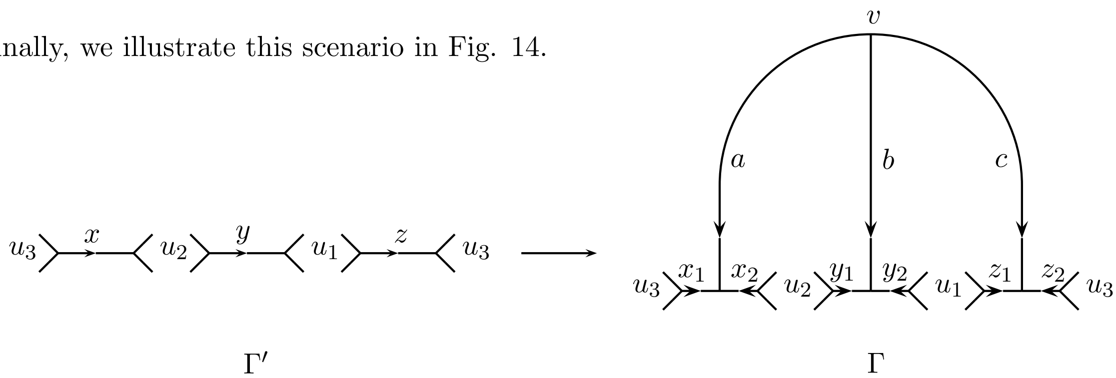


Fig. 14

Remark 2.4. We know that an α, β or γ construction must increase the genus by 1 as if we perform any of these constructions to a MOWF w of length $12g-6$, whose associated cubic graph has $4g-2$ vertices by Remark 2.1 then, we find a new MOWF w' whose associated graph has 4 extra vertices. Thus, the number of vertices $v = 4g-2+4 = 4g+2 = 4(g+1)-4+2 = 4(g+1)-2$, showing that g has necessarily increased by 1 in order to satisfy the relationship between the genus and the number of vertices.

Definition 2.4. If a MOWF w of genus g can be obtained from a MOWF w' of genus $g-1$ by either an α, β or γ-construction, then we say w' is the *reduction* of w with respect to the negative vertex v of type α, β or γ. We call these reductions α*,* β *or* γ *reductions.*

Consider the structure that must be present in the cubic graph Γ associated to w in order to perform either an α or β reduction. These reductions will always be paired as if there is a double edge between two vertices, they are necessarily both negative and of the same type (so they are both either type α or type β vertices). This implies that a reduction with respect to either vertex would give exactly the same result.

Previously, we concluded that all cubic graphs that give rise to MOWFs can be obtained by taking combinations of the genus 1 graph and type α, β or γ vertices. In other words, the α, β and γ constructions can be used recursively to find all MOWFs that have genus greater than 1.

Let $w = w_1 \dots w_{2l}$ be a MOWF of genus g (which implies $l = 6g - 3$). Suppose x is some edge of the cubic graph Γ related to w. We associate a transformation of w to this edge, called the *IH transformation with respect to x*. In terms of the graph Γ, the IH transformation with respect to x amounts to contracting x (or identifying its initial and terminal vertices) to form a new vertex of degree 4. It is then possible to split this new vertex in two ways. One of these ways will return the same graph, the other will produce a different graph that emits a different Bi-Eulerian path and hence, a different MOWF w', which is the result of the IH transformation with respect to x on w (see Fig. 15).

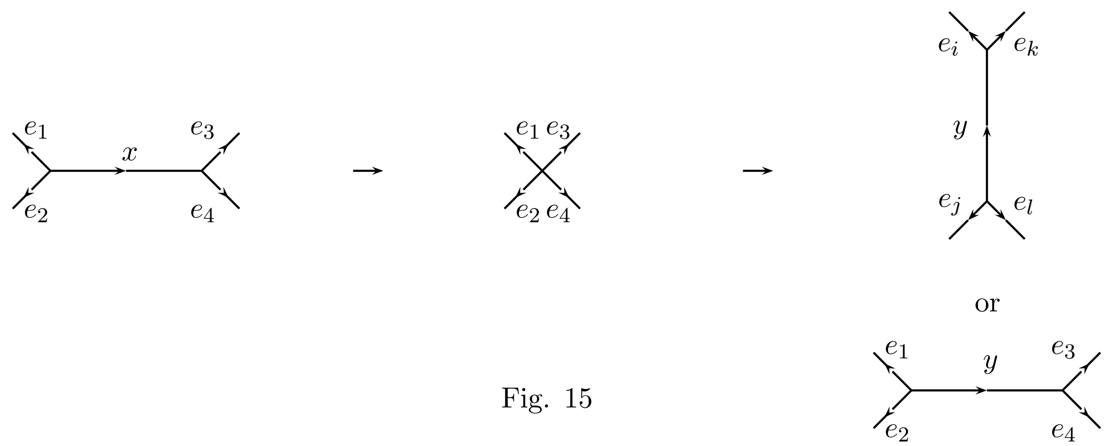


Fig. 15

Note that i, j, k and l are all one of the subscripts $1-4$. However, which value they take may vary due to the way the edges were traversed in the first place. The way to traverse the edges is encoded in the MOWF that is associated to the graph. As much of the MOWF will remain the same following an IH transformation with respect to x, it may be that certain edges need to be interchanged to ensure they can be still be traversed in the correct way.

The definition of an IH transformation with respect to x on a cubic graph Γ, associated to a MOWF, can be formalised by considering the subwords of Γ that describe how to traverse x and x^{-1}. To find a general form for these subwords we think of the local structure of Γ near x to be like Fig. 15. That is, edges e_1 and e_2 are leaving the initial vertex of x and e_3 and e_4 are leaving the terminal vertex of x. With this local structure in mind, we deduce that there are only four possible ways to traverse x that obey the rules set out in Definition 2.1. These possibilities imply that w contains one of the following pairs of subwords:

(i) $e_1^{-1}xe_4$ and $e_3^{-1}x^{-1}e_2$;

(ii) $e_1^{-1}xe_3$ and $e_4^{-1}x^{-1}e_2$;

(iii) $e_2^{-1}xe_4$ and $e_3^{-1}x^{-1}e_1$;

(iv) $e_2^{-1}xe_3$ and $e_4^{-1}x^{-1}e_1$.

As (ii)-(iv) can be obtained from (i) by a permutation of the labels on the edges adjacent to x, we may neglect these options and work solely with option (i).

By ensuring that a MOWF w that contains the subwords $e_1^{-1}xe_4$ and $e_3^{-1}x^{-1}e_2$ satisfies Definition 2.1 and Remark 2.2, we find that

(a) $e_1 \neq e_2$,

(b) $e_1 \neq e_4$,

(c) $e_2 \neq e_3$,

(d) $e_3 \neq e_4$,

(e) $(e_3^{-1}, e_2) \neq (e_1, e_4^{-1})$ if $g > 1$,

where, the notation used in (v) means the two pairs of edges are not equal. These inequalities can be shown easily by contradiction.

(a) Suppose $e_1 = e_2$. The subwords become $e_1^{1}xe_4$ and $e_3^{-1}x^{-1}e_1$. Hence, w would contain both $e_1^{-1}x$ and $x^{-1}e_1$, a contradiction to (iii) in Definition 2.1.

(b) Suppose $e_1 = e_4$. The first subword becomes $e_1^{-1}xe_1$. This would imply x was a loop in Γ, contradicting Remark 2.2.

(c) Suppose $e_2 = e_3$. The second subword becomes $e_2^{-1}x^{-1}e_2$. We have the same problem as in (b).

(d) Suppose $e_3 = e_4$. The subwords become $e_1^{-1}xe_3$ and $e_3^{-1}x^{-1}e_2$. As w would contain both xe_3 and $e_3^{-1}x^{-1}$ we face the same problem as in (a).

(e) Suppose $(e_3^{-1}, e_2) = (e_1, e_4^{-1})$ for $g > 1$. This implies we have the genus 1 graph; a contradiction.

This leaves three other options. We could have that $e_1 = e_3^{-1}$ is the only equality among the e_i's, $e_2 = e_4^{-1}$ is the only equality or, there are no equalities at all. We will say that each situation gives rise to a type of IH transformation with respect to x. The case in which there are no relationships will be called a *type* 1 *IH transformation with respect to* x. If $e_1 = e_3^{-1}$ we must perform a *type* $2a$ *IH transformation with respect to* x; whereas $e_2 = e_4^{-1}$ will imply a *type* $2b$ *IH transformation with respect to* x. Remembering that the inequalities (a)-(e) must be true in all cases, we examine what happens to w when a type 1, $2a$ or $2b$ IH transformation with respect to x is performed on the graph Γ relating to w.
Type 1: We have $e_1 \neq e_3^{-1}$ and $e_2 \neq e_4^{-1}$ so, w must contain the subwords $e_2^{-1}e_1$ and $e_4^{-1}e_3$, as there are no other ways to traverse the edges given the conditions. The IH transformation with respect to x on Γ acts as shown in Fig. 16.

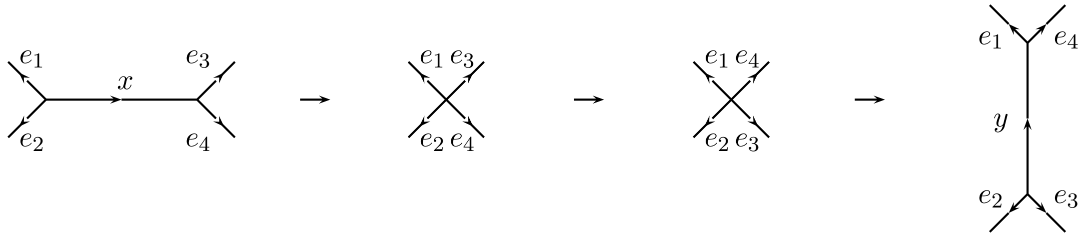


Fig. 16

By inspecting the subgraph we are left with, we deduce that the IH transformation with respect to x on Γ can be defined by making the following substitutions in w:

$$\begin{aligned} e_1^{-1}xe_4 &\mapsto e_1^{-1}e_4; \\ e_3^{-1}x^{-1}e_2 &\mapsto e_3^{-1}e_2; \\ e_2^{-1}e_1 &\mapsto e_2^{-1}ye_1; \\ e_4^{-1}e_3 &\mapsto e_4^{-1}y^{-1}e_3. \end{aligned}$$

Type $2a$**:** Here, $e_1 = e_3^{-1}$ meaning we know $e_4^{-1}e_1^{-1}xe_4$ and $e_2^{-1}e_1x^{-1}e_2$ are subwords of w. This is clear by inspection of the initial subgraph seen in Fig. 17.

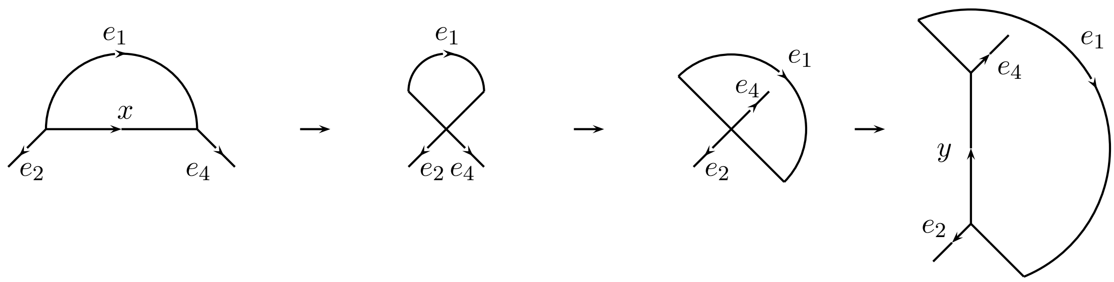


Fig. 17

Obeying the rules that were already set out in w, we have a specific way to traverse the new subgraph. Hence, in this case, the IH transformation with respect to x amounts to making the following substitutions in w:

$$\begin{aligned} e_4^{-1}e_1^{-1}xe_4 &\mapsto e_4^{-1}y^{-1}e_1^{-1}e_4; \\ e_2^{-1}e_1x^{-1}e_2 &\mapsto e_2^{-1}ye_1e_2. \end{aligned}$$

Type $2b$**:** We now suppose $e_2 = e_4^{-1}$. So $e_1^{-1}xe_2^{-1}e_1$ and $e_3^{-1}x^{-1}e_2e_3$ are subwords of w. We have Fig. 18.

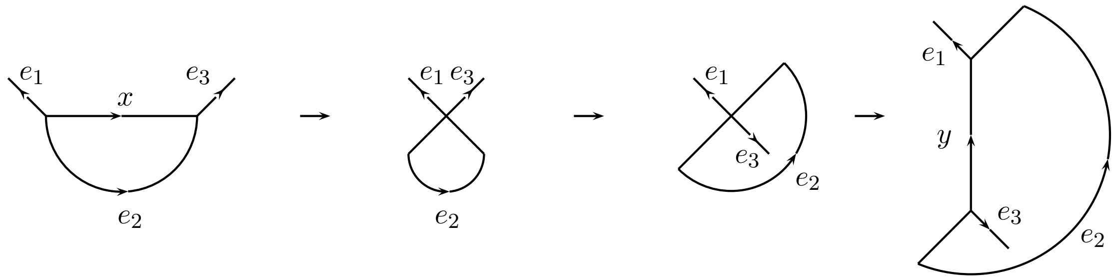


Fig. 18

Therefore, this type if IH transformation is the same as making the substitutions:

$$\begin{aligned} e_1^{-1}xe_2^{-1}e_1 &\mapsto e_1^{-1}e_2^{-1}ye_1; \\ e_3^{-1}x^{-1}e_2e_3 &\mapsto e_3^{-1}e_2y^{-1}e_3. \end{aligned}$$

Lemma 2.2. *(i) Performing IH transformations preserves MOWFs of genus g.*

(ii) Two MOWFs related by an IH transformation of type $2a$ or type $2b$ are equivalent.

Proof. (i) IH transformations preserve planarity of Γ on the surface S, obtained from the MOWF. Thus, the new MOWF that is obtained will have the same genus as the old MOWF; as both of these are embedded on S, a surface of genus g.

(ii) A type $2a$ IH transformation is described by the maps

$$\begin{aligned} e_4^{-1}e_1^{-1}xe_4 &\mapsto e_4^{-1}y^{-1}e_1^{-1}e_4; \\ e_2^{-1}e_1x^{-1}e_2 &\mapsto e_2^{-1}ye_1e_2. \end{aligned}$$

This amounts to a permutation of the labels on the edges of Γ. We have renamed the edge e_1 as y and the edge x as e_1, so the two MOWFs are equivalent. A similar argument works for type $2b$ transformations.

□

Lemma 2.3. *An α or β construction increases the number of positive and negative vertices by 2.*

Proof. This is easily proved by calling upon the our earlier work. Recall that a type α construction is related to Case N_2 (ii). When working through this case we observed that there will always be two positive vertices and two negative vertices present in the structure. The same was observed in Case N_2 (i), which relates to a type β construction. □

Lemma 2.4. *The number of positive or negative vertices is constant under IH transformations.*

Proof. This is clearly true for type $2a$ or type $2b$ IH transformations by Lemma 2.2 (ii). Let w and w' be two MOWFs. Suppose w is related to w' by a type 1 IH transformation with respect to x, where x is an edge in the graph Γ sitting on a surface S of genus g, obtained from w. Suppose w' is related to w by a type 1 IH transformation with respect to y, where y is an edge in the graph Γ' sitting on a surface S' of genus g, obtained from w. We have already seen that if an IH transformation of type 1 is possible then w must contain the subwords

$$e^{-1}xe_4,\ e_3^{-1}x^{-1}e_2,\ e_2^{-1}e_1 \text{ and } e_4^{-1}e_3;$$

whereas w' must contain the subwords

$$e_1^{-1}e_4,\ e_3^{-1}e_2,\ e_2^{-1}ye_1 \text{ and } e_4^{-1}y^{-1}e_3.$$

These sets of subwords must appear in the same cyclic order in each MOWF; everywhere else, the two MOWFs will be exactly the same. Hence, we only need to check that the six possibilities for the cyclic ordering of the subwords imply that both w and w' have the same number of positive and negative vertices. This amounts to inspecting the two vertices incident to x in Γ and the two vertices incident to y in Γ', as these are the only two vertices that might be altered by the IH transformations.

(1)

$$\begin{aligned} w &= e_1^{-1}xe_4\ldots e_3^{-1}x^{-1}e_2\ldots e_2^{-1}e_1\ldots e_4^{-1}e_3\ldots, \\ w &= e_1^{-1}e_4\ldots e_3^{-1}e_2\ldots e_2^{-1}ye_1\ldots e_4^{-1}y^{-1}e_3\ldots. \end{aligned}$$

In this case, x has one positive vertex and one negative vertex incident to it, as does y.

(2)

$$\begin{aligned} w &= e_1^{-1}xe_4\ldots e_3^{-1}x^{-1}e_2\ldots e_4^{-1}e_3\ldots e_2^{-1}e_1\ldots, \\ w &= e_1^{-1}e_4\ldots e_3^{-1}e_2\ldots e_4^{-1}y^{-1}e_3\ldots e_2^{-1}ye_1\ldots. \end{aligned}$$

Both x and y are incident to one positive and one negative vertex.

(3)

$$\begin{aligned} w &= e_1^{-1}xe_4\ldots e_2^{-1}e_1\ldots e_4^{-1}e_3\ldots e_3^{-1}x^{-1}e_2\ldots, \\ w &= e_1^{-1}e_4\ldots e_2^{-1}ye_1\ldots e_4^{-1}y^{-1}e_3\ldots e_3^{-1}e_2\ldots. \end{aligned}$$

Again, x and y are incident to one vertex of each orientation.

(4)

$$\begin{aligned} w &= e_1^{-1}xe_4\ldots e_2^{-1}e_1\ldots e_3^{-1}x^{-1}e_2\ldots e_4^{-1}e_3\ldots, \\ w &= e_1^{-1}e_4\ldots e_2^{-1}ye_1\ldots e_3^{-1}e_2\ldots e_4^{-1}y^{-1}e_3\ldots. \end{aligned}$$

Both x and y are incident to two negative vertices.

(5)

$$\begin{aligned} w &= e_1^{-1}xe_4\ldots e_4^{-1}e_3\ldots e_3^{-1}x^{-1}e_2\ldots e_2^{-1}e_1\ldots, \\ w &= e_1^{-1}e_4\ldots e_4^{-1}y^{-1}e_3\ldots e_3^{-1}e_2\ldots e_2^{-1}ye_1\ldots. \end{aligned}$$

We have that x and y are incident to two positive vertices.

(6)

$$\begin{aligned} w &= e_1^{-1}xe_4\ldots e_4^{-1}e_3\ldots e_2^{-1}e_1\ldots e_3^{-1}x^{-1}e_2\ldots, \\ w &= e_1^{-1}e_4\ldots e_4^{-1}y^{-1}e_3\ldots e_2^{-1}ye_1\ldots e_3^{-1}e_2\ldots. \end{aligned}$$

Finally, we are back to the case when x and y are incident to one negative and one positive vertex.

Thus, in each case, the graphs relating to w and w' must contain the same number of positive and negative vertices, as x and y are always incident to the same number of positive and negative vertices. □

Proposition 2.2. *A MOWF of genus g has exactly $2(g-1)$ positive vertices and $2g$ negative vertices.*

Proof. Suppose $g = 1$, then we have the genus 1 graph (or Case N_1), so the graph has no positive vertices and two negative vertices. Thus, the proposition holds for $g = 1$, as this implies $2(g-1) = 0$ and $2g = 2$.

We assume the result holds if $g = k$ and prove by induction. Suppose we have a MOWF w with $g = k+1$. Choose a loop λ in Γ, the graph related to w, that has minimal

length (i.e. we choose the shortest subpath of the Bi-Eulerian path on Γ that defines a loop). We can think of the possibilities for the length of λ as falling in to one of two cases.
Case 1: The loop λ has length 2. If this is true then there must be two doubly adjacent vertices in Γ. We already know that if there is a double edge between two vertices then they are necessarily both negative and of type α or β. Thus, performing either an α or β reduction with respect to one of the vertices joined by the double edge will produce a w' that has genus $g = k$. Reversing this process to get back to our original MOWF by performing an α or β construction would add two positive and two negative vertices by Lemma 2.3. So Γ has $2(k-1)+2 = 2((k+1)-1)$ positive vertices and $2k+2 = 2(k+1)$ negative vertices. Hence, by induction, the result is true in this case.
Case 2: The loop λ has length greater than 2. When we traverse λ we are forced to turn either left or right at every vertex we meet, giving us two options for how λ may be structured.

(a) The loop λ turns the same way at two consecutive vertices.

(b) The loop λ alternates the way that it turns at each vertex for its duration (i.e. if λ turns left at the first vertex it meets, it turns right at the second, left at the third and so on).

Suppose we find a λ that fits (a). So, λ turns the same way at the consecutive vertices v_i and v_{i+1}. Let x be the edge connecting v_i and v_{i+1} (we do not need to consider if there are multiple edges connecting the two vertices as if this was the case we would have a loop of length 2). Perform a type 1 IH transformation with respect to x. This produces a new graph Γ', corresponding to the MOWF w', that contains a shorter loop λ'. By Lemma 2.4, Γ and Γ' have the same number of positive and negative vertices so it is sufficient to work with Γ' from now on. If λ' has length 2 we are in Case 1 and hence, the result holds. If λ' has length greater than 2, then again we can say that the loop has the structure described in either (a) or (b). If the structure of λ' fits the description of (a), then we perform another type 1 IH transformation to obtain a second new graph with even shorter loop. We repeat this process until we have a loop of length 2, meaning the result is true; or we have a loop that matches the description of (b).

Suppose we find a λ that fits (b). Choose two vertices v_i and v_{i+1} traversed by λ and let x be the edge that connects them. Perform a type 1 IH transformation with respect to x. The new graph contains a shorter loop. Either this new loop has length 2, meaning the result holds; or the loop has length greater than 2. So, the loop fits either (a) or (b). If the loop fits (a) we are done. Otherwise, repeat the process obtaining new loops until a loop has length 2 or fits (a). ☐

The orientable Wicks forms $w = w_1w_2 \dots w_{2l}$ over the alphabet A and $w' = w'_1w'_2 \dots w'_{2l}$ over the alphabet A' are *isomorphic* if there is a bijection $\phi : A \to A'$ such that $\phi(w) =$

$\phi(w_1)\phi(w_2)\dots\phi(w_{2l}) = w'$.

Definition 2.5. The *automorphism group* Aut(w) of an orientable Wicks form $w = w_1w_2\dots w_{2l}$ over an alphabet A with genus g, is the group of cyclic permutations ρ of the word w such that w is isomorphic to $\rho(w)$ (i.e. $\rho(w)$ is obtained from w by permuting the letters of the alphabet A).

Note that Aut(w) is a subgroup of the group $\mathbb{Z}/2l\mathbb{Z}$ of all cyclic permutations of words of length $2l$.

Definition 2.6. The *mass* $m(W)$ of a finite set W of isomorphism classes of orientable Wicks forms as

$$m(W) = \sum_{w\in W} \frac{1}{|\mathrm{Aut}(w)|}.$$

We define the following sets:

- W_1^g is the set of all MOWFs of genus g (up to isomorphism);
- $W_2^g(r)$ is the set of all MOWFs that have an automorphism of order 2 such that r edges of the graph Γ arising from w are invariant, their orientation being reversed;
- $W_3^g(s,t)$ is the set of all MOWFs that have an automorphism of order 3 such that s positive and t negative vertices remain invariant in the Γ;
- $W_6^g(3r, 2s, 2t) = W_2^g(3r) \cup W_3^g(2s, 2t)$ is the set of all MOWFs that have an automorphism ρ order 6 such that ρ^3 leaves $3r$ edges invariant in Γ and ρ^2 leaves $2s$ positive and $2t$ negative vertices invariant in Γ.

Respectively, the masses of these sets are defined as:

- $$m_1^g = \sum_{w\in W_1^g} \frac{1}{|\mathrm{Aut}(w)|};$$
- $$m_2^g(r) = \sum_{w\in W_2^g(r)} \frac{1}{|\mathrm{Aut}(w)|};$$
- $$m_3^g(s,t) = \sum_{w\in W_3^g(s,t)} \frac{1}{|\mathrm{Aut}(w)|};$$
- $$m_6^g(3r; 2s, 2t) = \sum_{w\in W_6^g(3r;2s,2t)} \frac{1}{|\mathrm{Aut}(w)|}.$$

Note that $W_2^g(r)$, $W_3^g(s,t) \subset W_1^g$.

In each case, we have specified the attributes of the graph Γ related to a MOWF w that remain invariant under the automorphism. This is because if a MOWF w has an automorphism of given order, then so does the graph Γ. Automorphisms of graphs can be thought of as symmetries of graphs (as a graph automorphism preserves vertex connectivity), that are much easier to handle, especially as we restrict ourselves to connected cubic graphs. Thus, if a MOWF w has an automorphism of order 2, then so does the graph Γ related to this MOWF. A graph automorphism of order 2 can be realised as a reflection of Γ about the midpoints of some edges. Given any edge e in Γ, a reflection maps e to e^{-1} if and only if the reflection maps e^{-1} to e or; it maps e to an edge e' in Γ if and only if it maps e^{-1} to $(e')^{-1}$, e' to e and $(e')^{-1}$ to e^{-1}. So, the reflection will leave r edges invariant by reversing the orientation of these edges.

If a MOWF w has an automorphism of order 3, then this is seen in Γ as a rotational symmetry - specifically, it is a rotation by $120°$ about some vertices of Γ. So, the rotation maps an edge e in Γ to an edge e' in Γ if and only if e' maps to e'' when e'' maps to e (similarly, the inverses of these edges must cycle round in an analogous way). Thus, we must rotate about some number of vertices. Assuming s of the vertices we rotate around are positive and t are negative, then an automorphism of order 3 leaves s positive and t negative vertices invariant in Γ.

Suppose we have a MOWF w that has an automorphism of order 6, that we will think of as the graph automorphism ρ of Γ. As ρ has order 6 it must be a composition of a reflection in the midpoints of some edges and a rotation by $120°$ about some vertices of Γ. This only works if the edges that we reflect in are incident to the vertices we rotate about. So, we can reflect in either one or all three of the edges incident to any vertex we rotate about. We cannot reflect in two of the edges as this implies ρ has order 3. Clearly, as ρ has order 6, we may define automorphisms ρ^3 and ρ^2 of orders 2 and 3, respectively. As ρ amounts to rotating and reflecting Γ in the way defined above then, ρ^3 must leave $3r$ edges invariant (these will be all edges incident to any vertex that we rotate about). Similarly, ρ^2 must leave $2s$ positive and $2t$ negative vertices invariant, these will be the vertices we are rotating about.

At this point we are ready to state and prove the main result of this chapter, Bacher & Vdovina (2002).

Theorem 2.1. *(i) For every MOWF w, $Aut(w)$ is a cyclic group with order* 1, 2, 3 *or* 6.

(ii)

$$m_1^g = \frac{2}{1}\left(\frac{1^2}{12}\right)^g \frac{(6g-5)!}{g!\,(3g-3)!}.$$

(iii) $m_2^g(r) > 0$ *(with* $r \in \mathbb{N}$*) if and only if* $f = (2g+1-r)/4 \in \{0,1,2,\ldots\}$*, where*

$$m_2^g(r) = \frac{2}{2}\left(\frac{2^2}{12}\right)^f \frac{1}{r!}\frac{(6f+2r-5)!}{f!\,(3f+r-3)!}.$$

(iv) $m_3^g(s,t) > 0$ *if and only if* $f = (g+1-s-t)/3 \in \{0,1,2,\ldots\}$*,* $s \equiv 2g+1 \pmod 3$ *and* $t \equiv 2g \pmod 3$*, where*

$$m_3^g(s,t) = \frac{2}{3}\left(\frac{3^2}{12}\right)^f \frac{1}{s!\,t!}\frac{(6f+2s+2t-5)!}{f!\,(3f+s+t-3)!}$$

if $g > 1$ *and* $m_3^1(0,2) = \dfrac{1}{6}$.

(v) $m_6^g(3r;2s,2t) > 0$ *if and only if* $f = (2g+5-3r-4s-4t)/12 \in \{0,1,2,\ldots\}$*,* $2s \equiv 2g+1 \pmod 3$ *and* $2t \equiv 2g \pmod 3$*, where*

$$m_6^g(3r,2s,2t) = \frac{2}{6}\left(\frac{6^2}{12}\right)^f \frac{1}{r!\,s!\,t!}\frac{(6f+2r+2s+2t-5)!}{f!\,(3f+r+s+t-3)!}$$

if $g > 1$ *and* $m_6^1(3;0,2) = \dfrac{1}{6}$.

(vi) Set

$$\begin{aligned}
m_2^g &= \sum_{r\in\mathbb{N},(2g+1-r)/4\in\mathbb{N}\cup\{0\}} m_2^g(r),\\
m_3^g &= \sum_{s,t\in\mathbb{N},(g+1-s-t)/3\in\mathbb{N}\cup\{0\},s\equiv 2g+1 \pmod 3} m_3^g(s,t),\\
m_6^g &= \sum_{r,s,t\in\mathbb{N},(2g+5-3r-4s-4t)/12\in\mathbb{N}\cup\{0\},2s\equiv 2g+1 \pmod 3} m_6^g(3r,2s,2t)
\end{aligned}$$

and let M_d^g *be the number of equivalence classes of MOWFs of genus* g *that have an automorphism of order* d*. We have*

$$\begin{aligned}
M_1^g &= m_1^g + m_2^g + 2m_3^g + 2m_6^g,\\
M_2^g &= 2m_2^g + 4m_6^g,\\
M_3^g &= 3m_3^g + 3m_6^g,\\
M_6^g &= 6m_6^g
\end{aligned}$$

and $M_d^g = 0$ *if* d *is not a divisor of* 6.

Note that M_1^g is the number of inequivalent MOWFs of genus g.

Proof. (i) Suppose w is a MOWF that has associated graph Γ emitting an automorphism ρ of order d. If p is a prime that divides d then the automorphism $\rho^{d/p}$ will have order p.

Suppose $p \neq 2$, then $\rho^{d/p}$ does not have order 2 and hence, does not leave any invariant edges in Γ. Thus, the action of $\rho^{d/p}$ on Γ separates the edges of Γ into a number of p-cycles, meaning p must divide the number of edges in Γ; so $p|3(2g-1)$ for all g. Hence, $p = 3$.

Suppose $p \neq 3$, then $\rho^{d/p}$ does not have order 3 and hence, does not leave any invariant vertices in Γ. Thus, the action of $\rho^{d/p}$ on Γ separates the vertices of Γ into a number of p-cycles, meaning p must divide the number of vertices in Γ; so $p|2(2g-1)$ for all g. Hence, $p = 2$.

Thus, $d = 2^a 3^b$. We now let p be a prime power that divides d. Try $p = 2^2 = 4$. Then, the action of $\rho^{d/p}$ leaves no invariant edges or vertices implying p is both 2 and 3, a contradiction. Hence, $a \leq 1$.

A combination of the previous three paragraphs shows that $d = 2^a 3^b$, with $a \leq 1$. By noting that all orbits of ρ^{2^a} on the sets of positive or negative vertices have either 3^b or 3^{b-1} elements we see that $b \leq 1$. Hence, $d|6$, proving that any automorphism of a graph related to a MOWF must have order 1, 2, 3 or 6. As $\mathrm{Aut}(w)$ is known to be a group then it must be cyclic of order 1, 2, 3 or 6.

(ii) Any element of W_1^{g+1} can be found by performing either an α, β or γ construction to an element of W_1^g. Let w be a specific MOWF in W_1^g. There are $2\binom{6g-3}{1}$ ways to perform an α construction on w. This is because the graph Γ associated to w has $6g-3$ edges and we could attach the α construction to any of these edges. If we attach it to the edge e, then we can traverse the α construction after traversing e or after traversing e^{-1}. So, we have two possibilities for every choice of edge.

There are $4\binom{6g-3}{2} + 4\binom{6g-3}{1}$ ways to add a β construction, as a β construction has two distinct edges that need connecting to the graph and we can choose to attach both of these to the same edge of Γ or to two different edges. There are $\binom{6g-3}{1}$ ways to choose one edge to attach to and $\binom{6g-3}{2}$ ways to choose two edges to attach to. We can choose to traverse the β construction in 4 ways in each case.

There are $8\binom{6g-3}{3} + 16\binom{6g-3}{2} + 8\binom{6g-3}{1}$ ways to add a γ construction. We could choose to connect the three edges of the γ construction to either 1, 2 or 3 edges of Γ. If we connect to one edge, there are $\binom{6g-3}{1}$ ways to choose it. If we connect to two edges, there are $\binom{6g-3}{2}$ ways to choose them and two ways to decide which edge will have two ends of the γ construction connected to it. This gives $2\binom{6g-3}{2}$ different options. If we choose to connect to three edges, there are $\binom{6g-3}{3}$ ways to do this. In each case, there are 8 possible ways to traverse the γ construction.

Alternatively, let w be an element of W_1^{g+1}. By Proposition 2.2, w has $2(g+1)$ negative vertices. Hence, we can perform $2(g+1)$ different α, β or γ reductions to w (as there is a reduction with respect to each negative vertex) that would give $2(g+1)$ different elements of W_1^g.

The number of ways to pass from W_1^g to W_1^{g+1} and vice versa must coincide if we add suitable coefficients to the equation to account for two factors. The first of these being the fact that α and β constructions produce two negative vertices that then yield the same reduction. Secondly, we must account for the fact that applying the same construction to a MOWF w of genus g and a MOWF w' automorphic to w gives the same MOWF of genus $g+1$. Multiplying the number of ways to perform α and β constructions by 2 deals with the first issue. Let w be a MOWF of genus g. We have already shown that there are

$$2\binom{6g-3}{1}+4\binom{6g-3}{2}+4\binom{6g-3}{1}+8\binom{6g-3}{3}+16\binom{6g-3}{2}+8\binom{6g-3}{1}$$

ways to perform an α, β or γ construction to w. Any MOWF automorphic to w will give the same MOWFs of genus $g+1$ after performing each of the same constructions. So, to only count these once we will require the quantity

$$\left(2\binom{6g-3}{1}+4\binom{6g-3}{2}+4\binom{6g-3}{1}+8\binom{6g-3}{3}\right.$$
$$\left.+16\binom{6g-3}{2}+8\binom{6g-3}{1}\right)\frac{1}{|\mathrm{Aut}(w)|}.$$

Clearly, if we do this for all MOWFs of genus g and add them up we will be able to factor out the number of ways to perform the constructions and be left with

$$\left(2\binom{6g-3}{1}+4\binom{6g-3}{2}+4\binom{6g-3}{1}+8\binom{6g-3}{3}+16\binom{6g-3}{2}+8\binom{6g-3}{1}\right)m_1^g.$$

Including the multiplication by 2 we have the following equality

$$\left(4\binom{6g-3}{1}+8\binom{6g-3}{2}+8\binom{6g-3}{1}+8\binom{6g-3}{3}\right.$$
$$\left.+16\binom{6g-3}{2}+8\binom{6g-3}{1}\right)m_1^g=2(g+1)m_1^{g+1}.$$

Some simplification gives

$$\begin{aligned}
\left(20\binom{6g-3}{1}+24\binom{6g-3}{2}+8\binom{6g-3}{3}\right)m_1^g &= 2(g+1)m_1^{g+1},\\
2\left(5\cdot\frac{(6g-3)!}{1!\,6g-4!}+6\cdot\frac{(6g-3)!}{2!\,6g-5!}+2\cdot\frac{(6g-3)!}{3!\,6g-6!}\right)m_1^g &= (g+1)m_1^{g+1},\\
2\left(5(6g-3)+3(6g-3)(6g-4)+\frac{2(6g-3)(6g-4)(6g-5)}{6}\right)m_1^g &= (g+1)m_1^{g+1},\\
2(6g-3)\left(12g^2-\frac{1}{3}\right)m_1^g &= (g+1)m_1^{g+1},\\
2(2g-1)(36g^2-1)m_1^g &= (g+1)m_1^{g+1},\\
2(2g-1)(6g+1)(6g-1)m_1^g &= (g+1)m_1^{g+1}.
\end{aligned}$$

Set

$$m_1^g = 2\frac{(6g-5)!}{12^g g!\,(3g-3)!}.$$

We can easily show that this satisfies the recurrence relation. To do this start with

$$m_1^{g+1} = 2\frac{(6g+1)!}{12^{g+1}(g+1)!\,(3g)!}.$$

According to the recurrence relation, we also have that

$$m_1^{g+1} = \frac{2(6g+1)(6g-1)(2g-1)}{(g+1)}\left[\frac{2(6g-5)!}{12^g g!\,(3g-3)!}\right].$$

If the two formulas for m_1^{g+1} are equated, one can show that they are equivalent by elementary manipulation. Thus, as

$$m_1^g = 2\frac{(6g-5)!}{12^g g!\,(3g-3)!}$$

satisfies the recurrence relation and it is easily checked that the formula implies $m_1^1 = \frac{1}{6}$, then we have proved part (ii) by induction.

(iii) Let w be a MOWF whose associated graph Γ has an automorphism μ of order 2 that leaves exactly r edges invariant. By Remark 2.1, w has $6g-3$ edges. At most $2g+1$ of these edges can be left invariant. This gives two cases to consider; either $r < 2g+1$ or $r = 2g+1$.

Case 1: Suppose $r < 2g+1$, then $f = (2g+1-r)/4$ is greater than or equal to 1. Under μ, there are $(6g-3-r)/2$ orbits of edges that are not invariant. Consider the graph Γ/μ, where all edges invariant under μ have been removed from Γ. If we also remove all leaves and vertices of degree 2 we are left with a new graph $\tilde{\Gamma}$, that corresponds to a new MOWF $\tilde{w}$. This new graph will have $(6g-3-r)/2-r = 3(2g-r-1)/2)$ edges, thus having genus $f = (2g+1-r)/4 \geq 1$.

More formally, suppose $w = w_1 w_2 \dots w_{12g-6}$. The subword $w = w_1 w_2 \dots w_{6g-3}$ will contain exactly one representative from each of the orbits under the action of μ on Γ. We remove all letters w_k from the subword such that $w_{k+6g-3} = w_k^{-1}$. Call the new word, with length $6g-3-r$, w'. Within this word, we know that if w_k appears, then either w_k^{-1} or w_{k+6g-3}^{-1} will also appear exactly once. So, by replacing each w_{k+6g-3}^{-1} by w_k^{-1} we produce a word satisfying (i) of Definition 2.1. Next, we find a new word, w'' from w', by removing all cyclic subfactors of the form $w_k w_k^{-1}$. Clearly, w'' satisfies (ii) of Definition 2.1. Finally, we find $\tilde{w}$ by cancelling w_i (or w_j) with its inverse if $w_i w_j$ and $w_j^{-1} w_i^{-1}$ both appear in w'' as cyclic subfactors. Our final word $\tilde{w}$ now satisfies (iii) of Definition 2.1 and is hence a MOWF with genus f (this can be shown by counting how many edges are removed throughout the process in relation to the number of invariant edges). An example will illustrate this more clearly.

Example 2.3. The word $w = abcdea^{-1}fb^{-1}e^{-1}ghc^{-1}f^{-1}ig^{-1}d^{-1}h^{-1}i^{-1}$ is a MOWF of genus 2. It has an automorphism μ of order 2 such that $\mu(abcdea^{-1}fb^{-1}e^{-1}) = ghc^{-1}f^{-1}ig^{-1}d^{-1}h^{-1}i^{-1}$. As the only edge invariant under μ is c we have $r = 1$, $g = 2$ and $f = 1$. Note that the subword $abcdea^{-1}fb^{-1}e^{-1}$ contains exactly one representative from each orbit under the action of μ. Following the steps we described above, we must remove c and replace f by d^{-1} (as $\mu(f) = d^{-1}$). This leaves us with $abdea^{-1}d^{-1}b^{-1}e^{-1}$. We must cancel d and its inverse as both bd and $d^{-1}b^{-1}$ appear in the word as cyclic subfactors. We have $\tilde{w} = abea^{-1}b^{-1}e$, which is the unique MOWF of genus 1.

Any MOWF $\tilde{w}$ that is found using this process will posses some extra structure. Take the word w, from which $\tilde{w}$ has been formed, and write it counter clockwise along two concentric circles related by r radial segments, so that the radial segments are labelled by the edges invariant under μ. Suppose $\tilde{l}$ is a letter of $\tilde{w}$ whose pre-image is the letter l of w. We define a function ϕ on the set of letters within $\tilde{w}$ as; $\phi(\tilde{l}) = 0$ if l and l^{-1} are on the same circle and $\phi(\tilde{l}) = 1$ otherwise. We can check that ϕ will be well-defined and satisfies

$$\phi(e_1) + \phi(e_2) + \phi(e_3) \equiv 0 \pmod 2$$

whenever e_1, e_2 and e_3 are 3 edges incident to a common vertex in $\tilde{\Gamma}$. A function defined is this way is called a $\mathbb{Z}/2\mathbb{Z}$-*flow* on the graph $\tilde{\Gamma}$.

Suppose we have a MOWF $\tilde{w}$ with genus $f = (2g+1-r)/4$ that has an associated graph $\tilde{\Gamma}$ that emits a $\mathbb{Z}/2\mathbb{Z}$-flow, ϕ. We can form

$$\frac{(12f-6)(12f-2)\dots(12f-10+4r)}{r!}$$

MOWFs with genus g from $\tilde{w}$ that have an automorphism ρ of order 2. This is because there are $(12f-2)$ ways to attach the first invariant edge, $(12f-2)$ ways to attach the second and so on, with $(12f-10+4r)$ ways to attach the r^{th} edge. Then, we divide by $r!$ as this is the number of ways to order the invariant edges. Lastly, we use the $\mathbb{Z}/2\mathbb{Z}$-flow as a blueprint for gluing together the pre-images of the orbits of ρ.

The set of all $\mathbb{Z}/2\mathbb{Z}$-flows forms a vector space with dimension $2f$ over $\mathbb{Z}/2\mathbb{Z}$. So

$$2^{2f}\frac{(12f-6)(12f-2)\dots(12f-10+4r)}{r!}m_1^f = 2m_2^g(r),$$

where a factor of 2 has been added on the RHS to accommodate for the Wicks forms contributing to m_1^f that have weight 1 while they have weight $\frac{1}{2}$ in $m_2^g(r)$. We see that

$$m_1^f = 2\frac{(6f-5)!}{12^f f!\,(3f-3)!},$$

by part (ii). Then, the equation is satisfied by setting

$$m_2^g(r) = \frac{(6f+2r-5)!}{3^f r!\,f!\,(3f+r-3)!},$$

proving (iii) in this case.

Case 2: Suppose $f=0$. As we did in the first case, we think about the ways to form MOWFs of genus g. The first invariant edge must be attach to the empty word. For the second and third edges there are 2 ways to attach them, for the forth there are 6 possibilities and so on. Since an empty graph cannot have a $\mathbb{Z}/2\mathbb{Z}$-flow then we have

$$2m_2^g(2g+1) = 2\frac{2\cdot 6\cdots(4r-10)}{r!}.$$

Simple substitution shows that this works.

(iv) **Case 1:** Suppose $t>0$ and that w is a MOWF of genus g that has an automorphism of order 3, ρ, fixing s positive and t negative vertices. For each of the fixed vertices there is a corresponding reduction (of type α, β or γ). Each of these reductions will produce an oriented Wicks form w' with genus $g-1$ that has an automorphism of order 3, ρ', fixing $t-1$ negative and s positive vertices.

Conversely, suppose we have an oriented Wicks form w' of genus $g-1$ that has an automorphism of order 3, ρ', fixing $t-1$ negative and s positive vertices. There are $2(2g-3)$ γ-constructions that can be performed to obtained a Wicks form w with genus g that has an automorphism of order 3, ρ, fixing t negative and s positive vertices. Thus,

$$2(2g-3)m_3^{g-1}(t-1,s) = tm_3^g(s,t).$$

As we have done before we show that this equation is also satisfied by the equation given in statement (iv) of the theorem and we have the result.

Case 2: Suppose $t = 0$, so there are no negative vertices that remain invariant under some automorphism ρ. We can conclude that $g > 1$ as $g = 1$ implies there are only two vertices, both of which are negative, so we would have no invariant vertices of any type. Let w be our MOWF of genus g that has an automorphism ρ of order 3, fixing s positive vertices and no negative vertices. As ρ does not leave any invariant edges, there are $(6g-3)/3 = 2g-1$ orbits of edges. Now, if $g > 1$, invariant edges can never be adjacent under ρ. Thus, there are s orbits of edges incident to the same vertex fixed by ρ. If these orbits are removed, we produce a graph on the orbit space having s vertices of degree 2. We then remove these vertices of degree 2 to find a MOWF $\tilde{w}$ of genus $f = (g+1-s)/3$, where all vertices have degree 3, there is one face and $2g-1-2s = 6f-3$ edges. The construction we have just performed is analogous to the construction described in the proof of part (ii). However, in this case, the graph would consist of 3 concentric circles connected by s radial segments (representing invariant positive vertices and their edges). Again, this MOWF will posses some extra structure. This time, the graph has a $\mathbb{Z}/3\mathbb{Z}$-flow ϕ such that $\phi(e) \equiv -\phi(e^{-1}) \pmod 3$ and $\phi(e_1) + \phi(e_2) + \phi(e_3) \equiv 0 \pmod 3$, whenever e_1, e_2 and e_3 are edges pointing towards a common vertex.

Conversely, suppose $\tilde{w}$ is a MOWF of genus f whose graph has an associated $\mathbb{Z}/3\mathbb{Z}$-flow. There are

$$\frac{(12f-6)(12f-2)\cdots(12f-10+4s)}{s!}$$

ways to form a MOWF w of genus g that has an automorphism ρ of order 3, fixing s positive and no negative vertices.

Again, the set of all $\mathbb{Z}/3\mathbb{Z}$-flows forms a vector space over $\mathbb{Z}/3\mathbb{Z}$ with dimension $2f$. Thus,

$$3^{2f}\frac{(12f-6)(12f-2)\cdots(12f-10+4s)}{s!}m_1^f = 3m_3^g(s,0) = 3m_3^{3f+s-1}(s,0).$$

The equation is easily shown to be satisfied if we set

$$m_1^f = 2\frac{(6f-5)!}{12^f f!\,(3f-3)!}$$

and

$$m_3^{3f+s-1}(s,0) = \frac{2}{3}\left(\frac{3}{4}\right)^f \frac{(6f+2s-5)!}{s!\,(3f+s-3)!\,f!},$$

proving (iv) if $f \geq 1$. The proof for $f = 0$ is analogous to the proof of this case in part (iii).

(v) Let w be a MOWF with genus g that has an automorphism of order 6, ρ. Consider the automorphism ρ^3 of order 2. Applying the reduction used in the proof of part (iii) we obtain a MOWF $\tilde{w}$ with genus $h = (2g+1-3r)/4$ whose associated graph has a $\mathbb{Z}/2\mathbb{Z}$-flow ϕ. As $\tilde{w}$ is in $W_3^h(s,t)$ it has an automorphism $\tilde{\rho}$ of order 3 that leaves ϕ invariant. Analogously to the proof of part (iii), we can use this information to find MOWFs in $W_6^g(3r; 2s, 2)$ by ensuring that all constructions are $\tilde{\rho}$-invariant. We require the following lemma that outlines the vector space of $\tilde{\rho}$-invariant $\mathbb{Z}/2\mathbb{Z}$-flows.

Lemma 2.5. *Let $\tilde{w}$ be an element of $W_3^h(s,t)$ with automorphism $\tilde{\rho}$ of order 3, fixing t negative and s positive vertices. Then, the vector space of all $\tilde{\rho}$-invariant $\mathbb{Z}/2\mathbb{Z}$-flows on $\tilde{\Gamma}$ has dimension $(h+1-s-t)/3$.*

Proof. Let $\tilde{\phi}$ be a $\tilde{\rho}$-invariant $\mathbb{Z}/2\mathbb{Z}$-flow on $\tilde{\Gamma}$. If e_1, e_2 and e_3 are three edges incident to a common $\tilde{\rho}$-invariant vertex then $\rho(\tilde{e_1}) \equiv \rho(\tilde{e_2}) \equiv \rho(\tilde{e_3}) \equiv 0 \pmod 2$. So, all the reductions performed in the proof of part (iv) can be applied here. As they are injective on $\tilde{\rho}$-invariant $\mathbb{Z}/2\mathbb{Z}$-flows the lemma is proved. □

Thus,

$$3^{(h+1-s-t)/3}\frac{(4h-6)(4h-2)\cdots(4h-10+4r)}{r!} m_3^h(s,t) = 2m_6^g(3r; 2s, 2t)$$

and following the same steps as we have previously we have the result.

□

Chapter 3

Subgroups of the Modular Group

Recall from the introduction that the Modular Group can be thought of as the free product of the cyclic group of order 2 and the cyclic group of order 3. This led to a presentation for the Modular Group of $M = \langle m_1, m_2 | m_1^2 = m_2^3 = e \rangle$, where we use e to represent the identity element of M. From this, we inferred that coset diagrams of subgroups of M will be made up of edges labelled m_1 that form cycles of length dividing 2 and edges labelled m_2 that form cycles of length dividing 3; motivating the first definition of the next section.

3.1 Counting Free Subgroups of the Modular Group with a Recurrence Relation

Definition 3.1. An *n-configuration* consists of n vertices, labelled $1, \ldots, n$, together with directed edges (including loops) coloured either red or green such that:

(i) each vertex has one edge of each colour entering and one edge of each colour leaving;

(ii) the red edges form cycles whose lengths divide 2 and the green edges form cycles whose lengths divide 3;

(iii) the entire diagram is connected.

The red and green cycles in (ii) will be called *degenerate* if they are formed from less than 2 or 3 edges, respectively. An n-configuration that has no degenerate cycles will be referred to as a *non-degenerate n-configuration.* We say two n-configurations are equivalent if one can be obtained from the other by permuting all but one vertex that has been selected and labelled as *special.*

Remark 3.1. The special vertex plays the role of the trivial coset of this subgroup and we introduce an equivalence relation due to the need to quotient out diagrams describing the same subgroup, where the cosets of the subgroup have been enumerated in a different order.

Proposition 3.1. *There is an equivalence relation $\sim_P$ among n-configurations obtained by permuting the non-special vertices.*

Proof. We refer to the set of all n-configurations as $\mathcal{D}_n$. For each D_{n_i} in $\mathcal{D}_n$ we choose the vertex labelled 1 as the special vertex. We know that for any D_{n_1}, D_{n_2} in $\mathcal{D}_n$, $D_{n_1} \sim_P D_{n_2}$ if and only if D_{n_1} can be obtained from D_{n_2} by permuting all but the special vertex. Obviously, this relation is reflexive and symmetric. Transitivity follows as the composition of two permutations of any given set is closed in relation to that set. ☐

Definition 3.2. Take the set of n symbols $X_n = \{1, \ldots, n\}$ and define S_n be the set of all permutations of X_n . Let (t, s) be a pair of elements in S_n. We say (t, s) is a *legitimate pair* if:

(i) $t^2 = s^3 = 1$;

(ii) the group generated by t and s is transitive.

Two legitimate pairs are defined to be equivalent if there exists an element r of S_n, fixing the symbol 1, such that conjugation by r sends one pair to the other. So (t_1, s_1) and (t_2, s_2) are equivalent if there exists $r \in S_n$ such that $(rt_1r^{-1}, rs_1r^{-1}) = (t_2, s_2)$.

Lemma 3.1. *Every equivalence class of n-configurations gives rise to a unique equivalence class of legitimate pairs.*

Proof. Take any n-configuration D_{n_i}. The red edges of D_{n_i} define a permutation t of the vertices $1, \ldots, n$ of D_{n_i} such that $t^2 = 1$. The green edges define a permutation s of the vertices such that $s^3 = 1$. Clearly, both t and s are elements of S_n. Recall that a group G acting on a set Y is transitive if its group action is transitive, i.e. has a single orbit, where for all y in Y

$$Orbit(y) = \{hy \in Y | h \in H\}.$$

We form the group F generated by t and s. There is an obvious action of F on the set X_n. This action has a single orbit if for every pair of elements x, y in X_n, there is an f in F such that $fx = y$. We see that such an f exists for any pair of elements by observing that F is generated by permutations describing disjoint sets of edges of a connected graph, where the union of these disjoint sets gives us the whole edge set of the graph. Subsequently, the elements of F are all possible paths in D_{n_i}. So, as the diagram is connected, there must exist a path from x to y and hence, an f in F such that $fx = y$. Thus, the group generated by t and s is transitive, meaning (t, s) is a legitimate pair.

Take any equivalence class of n-configurations. Let D_{n_1} and D_{n_2} be two graphs in this equivalence class and let (t_1, s_1) and (t_2, s_2) be the two legitimate pairs arising from D_{n_1} and D_{n_2}, respectively. As D_{n_1} and D_{n_2} are in the same equivalence class one can be obtained from the other by a permutation, r, of the vertices. This r serves as the element of S_n that sends (t_1, s_1) to (t_2, s_2) by conjugation. Thus, any equivalence class of n-configurations in fact defines an equivalence class of legitimate pairs. □

Lemma 3.2. *Let H be a subgroup of M with index n and enumerate the cosets of H as $B_1, \ldots, B_n$. There exists j such that $m_i B_j = B_j$ for some i, if and only if H is not free.*

Proof. Suppose H is free and there exists i, j such that $m_i B_j = B_j$. As B_j is a coset of H then $B_j = mH$ for some m in M. We have two cases to consider, either m in H or m not in H.
Case 1: If m in H then

$$m_i B_j = B_j \Rightarrow m_i m H = mH \Rightarrow m_i H = H.$$

So, for some h in H we have $m_i h = e$. Hence, $h = m_i^{-1}$ and so m_i in H.
Case 2: If m not in H then

$$m_i B_j = B_j \Rightarrow m_i m H = mH \Rightarrow m^{-1} m_i m H = H.$$

So, for some h in H we have $m^{-1} m_i m h = e$. Hence, $h = m^{-1} m_i^{-1} m$ and so $h^{p_i} = e$, where $p_1 = 2$ and $p_2 = 3$.

In both cases we have shown that H contains an element of finite order and therefore is not free. □

Theorem 3.1. *There is a one-to-one correspondence between equivalence classes of n-configurations and subgroups of index n in M. In particular, the equivalence classes of non-degenerate n-configurations of order n correspond to the free subgroups of index n in M.*

Proof. Suppose we have some subgroup H of M with index n. We can enumerate the cosets of H as $B_1, \ldots, B_n$, where we choose B_1 to represent the coset H. The cosets of H partition M, so $m_i B_u = B_v$ for some u, v. Hence, if we compose a coset of H with a generator of M the result is also a coset of H.

Take a graph with vertices labelled $1, \ldots, n$, where the vertex labelled j corresponds to the coset B_j. We add a red edge (or loop) between two vertices corresponding to cosets B_u, B_v if $m_1 B_u = B_v$ (or at a vertex corresponding to B_w if $m_1 B_w = B_w$). Similarly, we add a green edge (or loop) between two vertices corresponding to cosets B_u, B_v if $m_2 B_u = B_v$ (or at a vertex corresponding to B_w if $m_2 B_w = B_w$). Clearly, each vertex has one edge of each colour entering and one leaving. These edges are given the obvious orientation.

Set $p_1 = 2$, $p_2 = 3$ and let c_1 be the colour red and c_2 be the colour green. Consider the effect that the generator m_i of M will have on the cosets of H. This generator will separate the cosets in to cycles of colour c_i that have lengths l_j, where l_j divides the order p_i of the generator m_i for all j. This is easily shown by contradiction. We assume there exists a cycle of colour c_i for $i = 1$ or $i = 2$, that has length l such that $l \nmid p_i$. Then, $m_i^l B = B$ for some coset B of H. We note that it is possible to consider what happens as two separate cases where either $l < p_i$ or $l > p_i$.

Case 1: If $l < p_i$ we have $p_i = al + b$ for some integers a, b with $b < l$. Hence,

$$B = m_i^{p_i} B = m_i^{b+al} B = m_i^b m_i^{al} B = m_i^b B \neq B.$$

Case 2: If $l > p_i$ we have $l = xp_i + y$ for some integers x, y with $y < p_i$. Hence,

$$B = m_i^{y+xp_i} B = m_i^y m_i^{xp_i} B = m_i^y B.$$

As $y < p_i$ and $y \nmid p_i$ we are back to the first case and so we have proved our earlier statement. Thus, the graph is made up of red cycles that have length dividing 2 and green cycles that have length dividing 3.

Coupling our knowledge that any element of M can be written as a combination of powers of the generators, with the fact that red and green edges have been added to the graph in accordance with the composition of cosets of H with generators of M, we see that it is possible to find a path made up of red and/or green edges from the vertex labelled 1 to any other vertex of the graph. Hence, the whole graph is connected and every subgroup of index n in G gives rise to a unique equivalence class of n-configurations, where all such graphs in the equivalence class are found by choosing a different ordering of the cosets in our enumeration of them.

To show that an equivalence class of n-configurations corresponds to a unique subgroup of index n in G we use Lemma 3.1 alongside the following theorem, Millington (1969); Stothers (1974) .

Theorem 3.2. *There is a one-to-one correspondence between legitimate pairs of S_n and subgroups of index n in M.*

We are left with the task of showing that it is the equivalence classes of non-degenerate n-configurations that correspond to the free subgroups of M. As the cycles formed by the red and green edges must divide 2 and 3, respectively, we see that the only non-degenerate cycles that can occur of either colour are of length 1; i.e. they are loops. A diagram of order n contains a loop if and only if $m_i B_j = B_j$ for some generator m_i of M and some coset B_j of H. Hence, applying Lemma 3.2 we obtain the result. ☐

By establishing this connection between n-configurations and subgroups of M it is possible to produce a recurrence relation to count the number of free subgroups of index n. First, we will require the following result.

Corollary 3.1. *If M has a free subgroup of index n, then $6|n$.*

Proof. As M has a free subgroup of index n, we know there exists a non-degenerate n-configuration. Hence, all red cycles have length 2 and all green cycles have length 3. By definition, each vertex has one edge of each colour entering and one leaving. Thus, every vertex is included in exactly one 2-cycle and exactly one 3-cycle. So, $2|n$ and $3|n$ so $6|n$. ☐

Hence, the index n of any free subgroup of M must have the form $n = 6k$ for some k in $\mathbb{N}$. Again, we set $p_1 = 2$ and $p_2 = 3$. We let $M(n)$ be the number of free subgroups of index n in G and begin to find a formula for $M(n)$ by counting the number of ways to separate $6k$ vertices into p_i-cycles (i.e. we count all the ways to form non-degenerate cycles). This will be denoted by $a_i(6k)$.

$a_i(6k)$ is the number of ways to split the vertices $1, \ldots, 6k$ into $l = 6k/p_i$ equal size sets. This can be done by repeatedly choosing p_i vertices until all vertices have been chosen. However, we must take into account the fact that choosing identical sets of vertices in different orders will be equivalent, so we must include division by $l!$ (the number of ways to order the l sets of vertices) in our formula for $a_i(6k)$. Moreover, there are $(p_i - 1)!$ ways to connect the p_i-cycle that each set of vertices represents, as we can choose the order that the vertices in the cycles appear in. Thus, we must also include multiplication by $[(p_i - 1)!\,]^l$ in our formula. This gives the following.

$$\begin{aligned}
a_i(kd) &= \frac{[(p_i-1)!\,]^l}{l!}\left[\binom{kd}{p_i}\cdot\binom{kd-p_i}{p_i}\cdot\ldots\cdot\binom{kd-(l-1)p_i}{p_i}\right],\\
&= \frac{[(p_i-1)!\,]^l}{l!}\left[\frac{(kd)!}{p_i!\,(kd-p_i)!}\cdot\frac{(kd-p_i)!}{p_i!\,(kd-2p_i)!}\cdot\ldots\cdot\frac{(kd-(l-1)!}{p_i!\,(kd-lp_i)!}\right],\\
&= \frac{[(p_i-1)!\,]^l}{l!}\cdot\frac{(kd)!}{(p_i!\,)^l},\\
&= \frac{(kd)!}{p_i^l l!}.
\end{aligned}$$

Set $e_i = 6/p_i$ and we have $l = ke_i$. Hence, our final formula is,

$$a_i(kd) = \frac{(kd)!}{p_i^{ke_i}(ke_i)!}.$$

Now, as the free subgroups are in one-to-one correspondence with the non-degenerate n-configurations, we can count the number of free subgroups of given index n in M by counting the number of non-degenerate n-configurations. This is done by finding all possible ways to connect $n = 6k$ vertices into p_1-cycles and p_2-cycles simultaneously.

Theorem 3.3. *Let*

$$A(6k) = \frac{a_1(6k) \cdot a_2(6k)}{(6k)!}.$$

For $k \geq 1$ we have,

$$M(6k) = 6kA(6k) - \sum_{j=1}^{k-1} A(6j)M(6(k-j)).$$

Proof. We can form $6k/p_1 = ke_1$ p_1-cycles from $6k$ vertices. Take a given set of ke_1 labelled and directed p_1-cycles. We count the $6k$-configurations which contain this set of p_1-cycles. Denote this by $T(6k)$. We can add a set of p_2-cycles to our set of p_1-cycles in $a_2(6k)$ ways, satisfying conditions (i) and (ii) of Definition 3.1, and producing no degenerate cycles. We only wish to consider those diagrams which are connected in order to satisfy condition (iii). The special vertex will be in a connected component that is itself a $6j$-configuration, for some $j \in \{1, .., k\}$ (as this configuration is made up of p_1-cycles and p_2-cycles and therefore its vertex set must have order divisible by 6). We count how many of these $a_2(6k)$ diagrams are disconnected; i.e. we count the number of ways to create a connected component that is a $6j$-configuration for $j \leq k-1$ that does not contain the special vertex. For any j, the number of ways this can happen is the number of ways to choose je_1 p_1-cycles from the $ke_1 - 1$ p_1-cycles that do not contain the special vertex, multiplied by the number of ways to add in the p_2-cycles to the $6j$-configuration we have created, multiplied by the number of ways to connect the other part of the diagram containing the special vertex. Hence, the number of disconnected diagrams will be

$$\sum_{j=1}^{k-1} \binom{ke_1 - 1}{je_1} a_2(6j)T(6(k-j)).$$

Thus,

$$T(6k) = a_2(6k) - \sum_{j=1}^{k-1} \binom{ke_1 - 1}{je_1} a_2(6j)T(6(k-j)).$$

We notice that $M(6k)$ is the number of equivalence classes containing $6k$-configurations that have no degenerate cycles, and that all $6k$-configurations containing no degenerate cycles will be in one of these classes. This gives us the following equation.

$$(6k-1)!\,M(6k) = a_1(6k)T(6k).$$

Hence,

$$l(6k-1)!\,M(6k) = a_1(6k)\left[a_2(6d) - \sum_{j=1}^{k=1}\binom{ke_1-1}{je_1}a_2(6j)T((6k-j))\right],$$

so

$$M(6k) = 6kA(6k) - \frac{6k}{p_1^{ke_1}(ke_1)!}\sum_{j=1}^{k-1}\frac{(ke_1-1)!}{(je_1)!\,(ke_1-1-je_1)!}a_2(6j)\frac{(6(k-j)-1)!}{a_1(6(k-j))}M(6(k-j)),$$

and

$$M(6k) = 6kA(6k) - \sum_{j=1}^{k-1}A(6j)M(6(k-j)).$$

□

Chapter 4

Main Result

Our main result is motivated by the work of Stothers. In particular, it is motivated by Theorem 3.3 which gives a recurrence relation to count free subgroups of index n in the Modular Group $M = C_2 * C_3$. By defining a second equivalence relation on the n-configurations it is possible to split them up into equivalence classes that are in one-to-one correspondence with conjugacy classes of subgroups of M. The n-configurations relating to free subgroups then lead to a collection of cubic graphs that correspond to MOWFs, allowing us to give a closed formula for the number of conjugacy classes of free subgroups in M of any given index n.

4.1 Conjugacy Classes of Subgroups of the Modular Group

Recall that we use $\mathcal{D}_n$ to denote the set of all n-configurations. Let $\mathcal{E}_n$ be a subset of $\mathcal{D}_n$ that contains one representative from each equivalence class of n-configurations, with respect to the equivalence relation $\sim_P$.

Observe that an n-configuration is a graph by definition. Hence, we can say two n-configurations are isomorphic if there exists a map between them that obeys the rules of a graph isomorphism.

Proposition 4.1. *There is an equivalence relation $\sim_I$ on $\mathcal{E}_n$ obtained by taking all isomorphisms of the n-configurations.*

Proof. Let $E_{n_1}, E_{n_2}, E_{n_3}$ be in $\mathcal{E}_n$. In each graph we choose the vertex labelled 1 as the special vertex. We know that $E_{n_i} \sim_I E_{n_j}$ if and only if E_{n_i} is isomorphic to E_{n_j}. Obviously, this relation is reflexive and symmetric. Transitivity follows as if $E_{n_1} \sim_I E_{n_2}$ and $E_{n_2} \sim_I E_{n_3}$ then there exists maps $f_1 : V(E_{n_1}) \to V(E_{n_2})$ and $f_2 : V(E_{n_2}) \to V(E_{n_3})$ that preserve vertex connectivity, where $V(E_{n_i})$ is used to denote the vertex set of the graph E_{n_i}. Thus, $f_3 = f_1 \circ f_2$ is a map from $V(E_{n_i})$ to $V(E_{n_3})$ that preserves vertex connectivity. Hence, $E_{n_1} \sim_I E_{n_3}$. □

Theorem 4.1. *There is a one-to-one correspondence between equivalence classes of* $\mathcal{E}_n$ *and conjugacy classes of subgroups of index* n *in* M*. In particular, the equivalence classes of* $\mathcal{E}_n$ *of non-degenerate* n*-configurations of order* n *correspond to the conjugacy classes of free subgroups of index* n*.*

Proof. If a group G acts on a set X then to each g in G we associate a map $\bar{g} : X \to X$, where $\bar{g} = x^g$ and x^g returns the action of g on x. It can be shown that the inverse of this map is $\overline{g^{-1}}$ and hence that $\bar{g}$ is a bijection. Thus, we can define a map

$$\rho : G \to Sym(X)$$
$$\rho : g \mapsto \bar{g}.$$

In fact, this is a group homomorphism of G into $Sym(X)$. All such group homomorphisms are referred to as *(permutation) representations* of G on X.

Let H be a subgroup of G and denote by $\mathcal{C}_H$ the set of all right cosets of H in G. The action of G on $\mathcal{C}_H$ gives a representation $\rho_H : G \to Sym(\mathcal{C}_H)$. We say two such representations ρ_H and ρ_K are equivalent if and only if $|\mathcal{C}_H| = |\mathcal{C}_K|$ and there is a bijection $\lambda : \mathcal{C}_H \to \mathcal{C}_K$ such that $\lambda(\alpha^{\rho_H(g)}) = (\lambda(\alpha))^{\rho_K(g)}$ for all α in $\mathcal{C}_H$ and for all g in G. It can be shown that ρ_H is equivalent to ρ_K if and only if H and K are conjugate in G, Dixon & Mortimer (1996).

Suppose H, K are subgroups of M, both with index n, and that their associated graphs D_H, D_K are in the same equivalence class with respect to the equivalence relation $\sim_I$, i.e. the graphs are isomorphic with $D_H = (\mathcal{C}_H, E_H)$ and $D_K = (\mathcal{C}_K, E_K)$. As both H and K have index n we know that $|\mathcal{C}_H| = |\mathcal{C}_K|$ and that we have permutation representations $\rho_H : M \to Sym(\mathcal{C}_H)$ and $\rho_K : M \to Sym(\mathcal{C}_K)$. As the graphs are isomorphic we know that there is an adjacency preserving map

$$\lambda : D_H \to D_K$$

such that if e_H in E_H with $e_H = (Hx^1_{e_H}, Hx^2_{e_H})$ and e_K in E_K with $e_K = (Kx^1_{e_K}, Kx^2_{e_K})$ for some $x^1_{e_H}, x^2_{e_H}, x^1_{e_K}, x^2_{e_K}$ in M then

$$\lambda(e_H) = e_K \iff \lambda(Hx^1_{e_H}) = Kx^1_{e_H} \text{ and } \lambda(Hx^2_{e_H}) = Kx^2_{e_K}.$$

Let $Hy \in \mathcal{C}_H$ and set $\lambda(Hy) = Kz$ for some $Kz \in \mathcal{C}_K$. Recall that the edges of D_H and D_K are labelled by either m_1 or m_2, where m_1 and m_2 are the generators of M such that $m_1^2 = m_2^3 = 1$. Let J be a subgroup of M with associated graph $D_J = (\mathcal{C}_J, E_J)$ and define P_J to be the set of all possible paths in D_J. Let p be in P_J, then $p = e^1_J e^2_J \dots e^s_J$ for some s in $\mathbb{N}$ where e^i_J in E_J for all i. Suppose each e^i_J has label m_{j_i} for j_i in $1, 2$. Then we define a map $\theta : P_J \to M$ by

$$\theta(e_J^i) = m_{j_i}.$$

Clearly, θ is a homomorphism so

$$\theta(e_J^1 e_J^2 \dots e_J^m) = m_{j_1} m_{j_2} \dots m_{j_m}.$$

Hence, any path in D_J can be seen as an element of M and any element of M can be seen as a path on D_J (as any x in M can be written in terms of the generators and we can find any string of generators we require by taking θ of some appropriate path in D_J).

Consider the structure of D_H around the vertex Hy and the structure of D_K around the vertex Kz (Fig. 19).

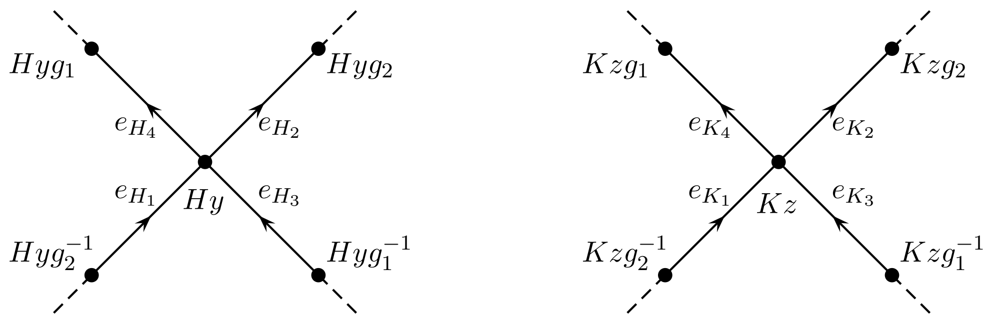


Fig. 19

By the definition of D_H, we know that $\theta(e_{H_1}) = \theta(e_{H_2}) = m_2$ and $\theta(e_{H_3}) = \theta(e_{H_4}) = m_1$. By the definition of D_K we know that $\theta(e_{K_1}) = \theta(e_{K_2}) = m_2$ and $\theta(e_{K_3}) = \theta(e_{K_4}) = m_1$. Hence, as λ is an adjacency preserving map, λ must map edges labelled m_i in D_H to edges labelled m_i in D_K.

Let e_H be some edge in D_H of the form $e_H = (Hy, Hy\theta(e_H))$. By definition $\lambda(e_H) = e_K$, where $e_K = (Kz, Kz\theta(e_K))$. Let e_H^{-1} be some edge of the form $e_H^{-1} = (Hy\theta(e_H^{-1}), Hy)$ in D_H. By definition $\lambda(e_H^{-1}) = e_K^{-1}$ where $e_K^{-1} = (Kz\theta(e_K^{-1}), Kz)$.

It is easy to see that λ also maps elements of P_H to P_K as λ is a homomorphism when considered as a map from E_H to E_K. Any element p_H of P_H has the form $p_H = e_H^1 e_H^2 \dots e_H^s$ so $\lambda(p_H) = \lambda(e_H^1 e_H^2 \dots e_H^s) = \lambda(e_H^1)\lambda(e_H^2)\dots\lambda(e_H^s) = e_K^1 e_K^2 \dots e_K^s$ for some e_K^i which is an element of P_K. Hence, $\lambda(p_H) = p_K$ for some p_K in P_K. So, $\theta(p_H) = \theta(p_K)$ as λ maps edges labelled m_i in D_H to edges labelled m_i in D_K.

By noting that $\lambda(\theta(p_H)) = \theta(\lambda(p_K))$ we have

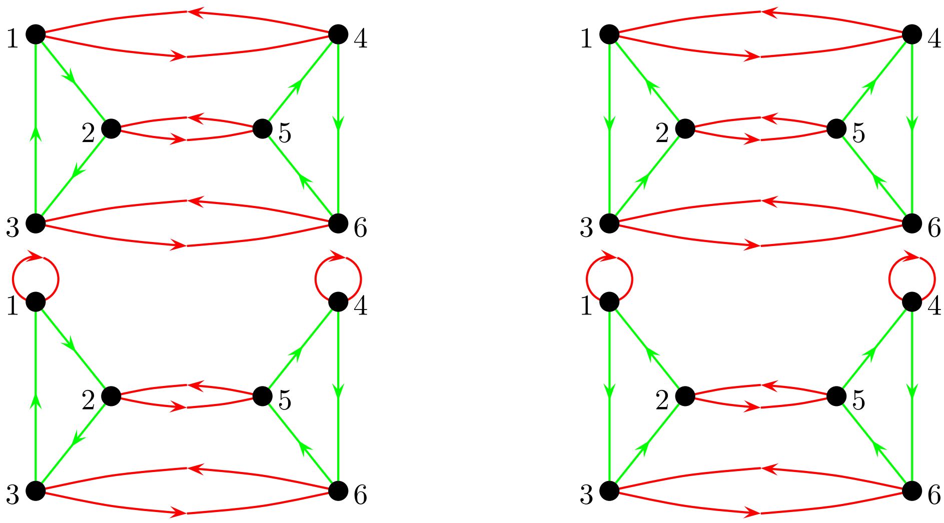
$$\begin{aligned}
\lambda(Hy^{(\rho_H(x))}) &= \lambda(Hyx) \\
&= \lambda(Hy\theta(p_H)) \\
&= \lambda(Hy)\lambda(\theta(p_K)) \\
&= kz\theta(\lambda(p_H)) \\
&= kz\theta(p_K) \\
&= kzx.
\end{aligned}$$

□

4.2 Counting Conjugacy Classes of Free Subgroups of the Modular Group

We now begin to focus on just those n-configurations which correspond to free subgroups. It will be useful to consider an example as we work through this section. To keep this simple we will use the most elementary case for which all results hold. Corollary 3.1 tells us that this is when $n = 6$.

Example 4.1. Let $n = 6k$ for some $k \in \mathbb{N}$. Then, the smallest possible n is 6. We find all graphs of order 6 to form $\mathcal{E}_6$, identifying the graphs of $\mathcal{E}_6$ which are isomorphic we are left with 8 equivalence classes. We give a representative of each in Fig. 20. All graphs in $\mathcal{E}_6$ can be found by taking all isomorphisms these graphs.



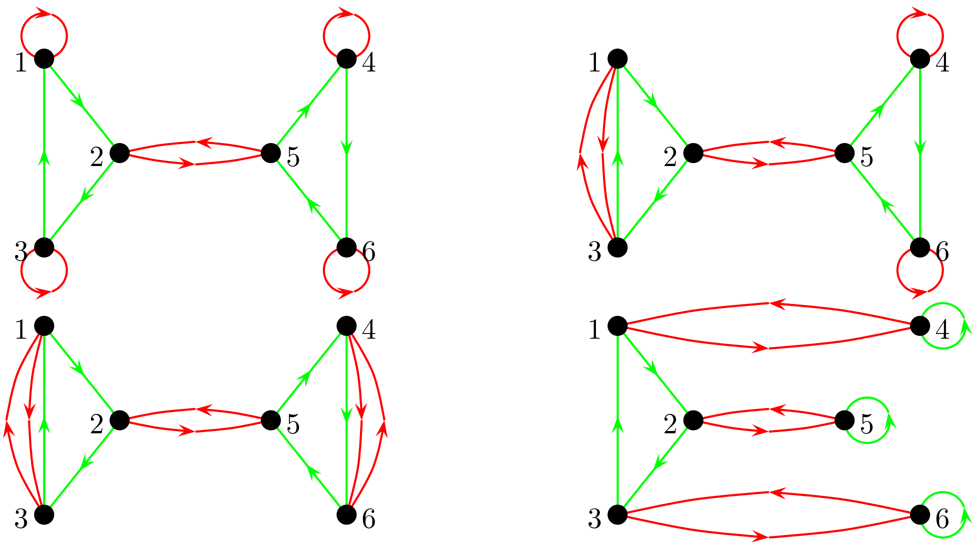


Fig. 20

Note that this tells us there are 8 conjugacy classes of subgroups of index 6 in M. The free subgroups correspond to the equivalence classes of non-degenerate n-configurations of order n. We notice that there are only 3 such graphs above. Hence, there are 3 conjugacy classes of free subgroups of index 6 in M.

We can find permutations A and B such that $A^2 = B^3 = 1$ related to each graph. We simply read this off the diagram, where A is defined by the red cycles and B is defined by the green cycles. The three graphs above that correspond to conjugacy classes of free subgroups give us the following pairs of cycles;

$$A_1 = (14)(25)(36) \text{ and } B_1 = (123)(546),$$
$$A_2 = (14)(25)(36) \text{ and } B_2 = (132)(546),$$
$$A_3 = (13)(25)(46) \text{ and } B_3 = (123)(546).$$

We will only be interested in the cases where $C = AB$ is transitive; i.e. when C is a single cycle over all the vertices of the graph. This occurs for only one of the above examples, when $C_2 = A_2B_2 = (162435)$. We remark that this means we are in fact only interested in those n-configurations that have an associated *finite transitive triple*. A finite transitive triple (Ω, A, B) is defined by a finite set Ω (where the cardinality of Ω is $|\Omega| = n$) and a pair of permutations A, B, where $A^2 = B^3 = 1$ and $C = AB$ is transitive on Ω, Brenner & Lyndon (1983).

Let D be a n-configuration. Let R be the set of red edges in D and let T be the set of green edges in D. All edges in R are either loops or part of a 2-cycle, by definition. All edges in T are either loops or part of a 3-cycle, by definition. We formulate an algorithm for the process of identifying n-configurations with associated transitive triples.

Algorithm 4.1.
1. Any loop in R has the form $e = (l, l)$ for some vertex l of D. Associate the 1-cycle (l) with an edge of this form.
2. Pair up edges e,e^{-1} in R, where $e = (i, j)$ and $e^{-1} = (j, i)$ for some vertices i and j of D. Associate the transposition (ij) with a pair of edges of this form.
3. Define $A = (l_1)(l_2)\ldots(l_r)(i_1j_1)(i_2j_2)\ldots(i_sj_s)$, where r is the number of loops in R and s is the number of 2-cycles in R. Then $A^2 = 1$.
4. Any loop in T has the form $e = (l, l)$ for some vertex l of D. Associate the 1-cycle (l) with an edge of this form.
5. Group together edges of the form e_1,e_2 and e_3 in T, where $e_1 = (i, j)$, $e_2 = (j, k)$ and $e_3 = (k, i)$ for some vertices i, j and k of D. Associate the 3-cycle (ijk) with a group of edges of this form.
6. Define $B = (l_1)(l_2)\ldots(l_u)(i_1j_1k_1)(i_2j_2k_2)\ldots(i_vj_vk_v)$, where u is the number of loops in T and v is the number of 3-cycles in T. Then $B^3 = 1$.

Take all n-configurations that have a related finite transitive triple. From each of these we can form a cubic graph of order $2k$ (a 3-regular graph with $2k$ vertices) with cyclic orders at the vertices. To do this we shrink the green 3-cycles to vertices with cyclic order and identify the edges of the red 2-cycles, i.e. we identify edges with their inverse.

Example 4.2. We continue to consider the graphs found in Example 4.1 for subgroups of index 6, showing in Fig. 21 how we obtain a cubic graph with cyclic orders at the vertices for the graph that has an associated finite transitive triple.

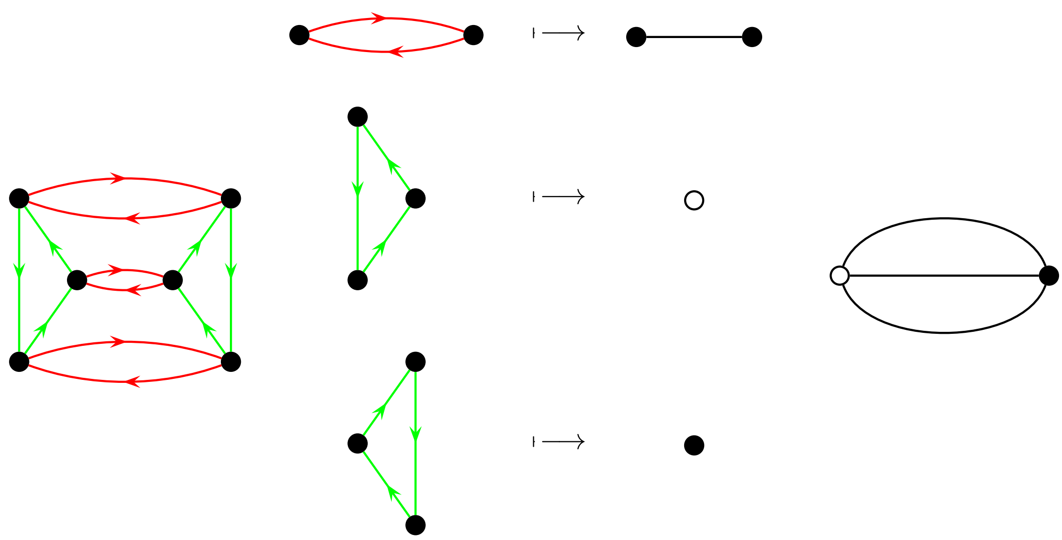


Fig. 21

Let D be a n-configuration of order n that gives permutations A and B (as described in Algorithm 4.1) such that $C = AB$ is transitive. Then, D has no loops of either colour. Let R be the set of red edges in D and let T be the set of green edges in D. All edges in R are part of a 2-cycle, by definition. All edges in T are part of a 3-cycle, by definition. We formulate the method of shrinking n-configurations into a general algorithm for later use.

Algorithm 4.2.

1. Take each pair of edges e, e^{-1} defining a 2-cycle in R and (ignoring orientation) identify these edges with each other to form a single undirected edge.

2. Take each group of edges e_1, e_2 and e_3 defining a 3-cycle in T. These edges have the form $e_1 = (i, j)$, $e_2 = (j, k)$ and $e_3 = (k, i)$ for some vertices i, j and k of D. Note that the vertices i, j and k are all adjacent to a unique undirected half edge (half of any undirected edge formed in (1)). Call these undirected half edges f_1, f_2 and f_3, where f_1 is the half edge adjacent to the terminating vertex of e_1, f_2 is the half edge adjacent to the terminating vertex of e_2 and f_3 is the half edge adjacent to the terminating vertex of e_3. Identify the vertices i, j and k to form a single vertex v and equip v with the cyclic orientation that takes f_1 to f_2, f_2 to f_3 and f_3 to f_1. We obtain a cubic graph of order $2k$ with cyclic orientation at vertices.

Any cubic graph of order $2k$ with cyclic ordering at the vertices that is found in this way will give us a MOWF. To obtain this we first find a cyclic Bi-Eulerian path on the graph. A *Bi-Eulerian path* on a graph is defined as a closed path that traverses every edge and its inverse exactly once, without traversing any edge followed immediately by its inverse. A *cyclic Bi-Eulerian path* on a graph is a Bi-Eulerian path where every cyclic permutation of the path is also a Bi-Eulerian path.

Example 4.3. It is possible to find such a path on the cubic graph of order $2k$ that we constructed in Example 4.2 (see Fig. 22). We must first assign an orientation to each edge in the graph and give each edge a unique label. Then, choose to begin at the white vertex and walk along A first. We obtain the Bi-Eulerian path $ac^{-1}ba^{-1}cb^{-1}$ and hence we obtain the orientable Wicks form w, where $w = ac^{-1}ba^{-1}cb^{-1}$.

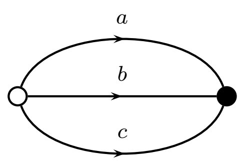


Fig. 22

Every cubic graph obtained in the way described above gives us a MOWF. We state this method formally. Let $\Gamma = (V, E)$ be a cubic graph of order $2k$ with cyclic orientation at vertices that has been constructed by Algorithm 4.2. Set $|E|= k$ and enumerate the edges of Γ as $e_1, e_2, \ldots, e_k$.

Algorithm 4.3. 1. Choose an orientation and a label to assign to each edge of Γ, this orientation and label is then fixed.

2. Select some vertex v of Γ and choose an edge e_i adjacent to v to traverse. If we traverse e_i is the direction of its orientation record e_i as the first letter of our orientable Wicks form. Otherwise, record e_i^{-1} as the first letter.

3. We are now at the some vertex u. We must traverse e_j, where the cyclic orientation at u sends e_i to e_j. If we traverse e_j is the direction of its orientation record e_i as the next letter of our orientable Wicks form. Otherwise, record e_j^{-1} as the next letter.

4. Repeat step (3) until we are forced to traverse e_i in the same direction as in step (1). The string of letters that has been recorded up to this point is a MOWF.

Lemma 4.1. *There is a one-to-one correspondence between cyclic Bi-Eulerian paths on cubic graphs of order* $2k$ *with cyclic orientation at the vertices (obtained by Algorithm 4.2) and MOWFs over the alphabet* $E \cup E^{-1}$.

Proof. Let p be a cyclic Bi-Eulerian path on some cubic graph of order $2k$ with cyclic orientation at the vertices. Call the graph $\Gamma = (V, E)$. Then, p traverses every edge and its inverse exactly once, satisfying (i) from Definition 2.1.

As p is cyclic Bi-Eulerian, every cyclic permutation of p does not traverse any edge followed immediately by its inverse. Hence, p also satisfies (ii) from Definition 2.1.

Now, suppose p traverses edge e_i followed by edge e_j and later traverses e_j^{-1} followed by e_j^{-1}. We deduce that $e_i = (v_r, v_s)$ and $e_j = (v_s, v_t)$, so both e_i and e_j are incident to v_s. As Γ is a cubic graph then v_s can only be adjacent to at most 3 unique edges. If v_s is adjacent to 3 unique edges let e_k be the remaining edge adjacent to v_s. Both of the other edges adjacent to v_s have been traversed in both directions. Hence, to traverse both e_k and e_k^{-1} we are forced to traverse them consecutively, giving a contradiction. If v_s is not incident to 3 unique edges, then there is a loop at v_s. If e_i is the loop at v_s we see that traversing e_i, e_j, e_i^{-1} and e_j^{-1} in the orders specified means v_s must be the initial and terminal vertex of p, hence p is not cyclic Bi-Eulerian. If e_j is the loop at v_s we see that traversing e_i, e_j, e_i^{-1} and e_j^{-1} in the orders specified means that these edges must form the following sequence $e_i e_j e_j^{-1} e_i^{-1}$ in p, contradicting our definition of p. Thus, (iii) from Definition 2.1 is satisfied. Thus, p in fact defines an orientable Wicks form over $E \cup E^{-1}$.

Finally, p is a MOWF over $E \cup E^{-1}$ as p traverses every edge and its inverse exactly once. Hence, p contains the whole of the alphabet $E \cup E^{-1}$.

Let $w = w_1 w_2 \ldots w_n$ be a MOWF over some alphabet A where $n = 12g - 6$ for some g in $\mathbb{N}$. We write w on the boundary of a disc as in Fig. 23.

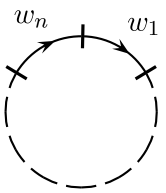


Fig. 23

We can identify the arcs of the disc (respecting orientation) to obtain an orientable compact surface S with embedded graph $\Gamma = (V, E)$. By definition, Γ must be a cubic graph of order $2k$. We equip Γ with cyclic orientation at the vertices using the following algorithm.

Algorithm 4.4. For each $v \in V$ do the following.

1. Suppose the edges entering v are e_1, e_2 and e_3 (where we take a^{-1} to be entering v if a is leaving v).
2. If w contains $e_1e_2^{-1}$, $e_2e_3^{-1}$, $e_3e_1^{-1}$ in any order then the orientation of v is given by (e_1, e_2, e_3).
3. If w contains $e_1e_3^{-1}$, $e_3e_2^{-1}$, $e_2e_1^{-1}$ in any order then the orientation of v is given by (e_3, e_2, e_1).

Hence, we have found a cubic graph of order $2k$ with cyclic orientation at the vertices.

□

Example 4.4. We show how it is possible to use the MOWF we found in Example 4.3 to construct a cubic graph of order $2k$ with cyclic orientation at the vertices.

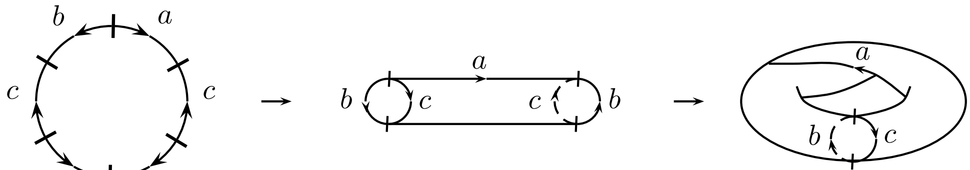


Fig. 24

We see that the Wicks form $w = ac^{-1}ba^{-1}cb^{-1}$ produces a torus with a twist; which is a surface of genus 1. The embedded graph Γ is Fig. 25, where we have applied Algorithm 4.4 to endow cyclic orientation on the vertices.

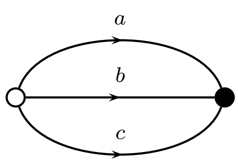


Fig. 25

Theorem 4.2. *There is a one-to-one correspondence between n-configurations with associated finite transitive triple and graphs of order $2k$ with cyclic orientation at the vertices that have a Bi-Eulerian path.*

Proof. Let D be a n-configuration that has an associated finite transitive triple (Ω, A, B). By definition, D has $3n/2$ edges. Assign a fixed labelling $e_1, e_2, \ldots, e_{3n/2}$ to the edges of D. The vertices of D are labelled 1 to n, so each edge has the form $e_i = (s, t)$ for s, t in $\{1, \ldots, n\}$. The permutation $C = AB$ defines a Bi-Eulerian path on D. Suppose $A = (a_1a_2)(a_3a_4)\ldots(a_{n-1}a_n)$ and $B = (b_1b_2b_3)(b_4b_5b_6)\ldots(b_{n-2}b_{n-1}b_n)$, where a_u, b_v are in $\{1, \ldots, n\}$, $a_{u_1} \neq a_{u_2}$ for all $u_1 \neq u_2$, $b_{v_1} \neq b_{v_2}$ for all $v_1 \neq v_2$ and for each a_u there exists v such that $a_u = b_v$.

The permutations A and B can be written as a list of maps as shown below.

$$\begin{array}{rcl} a_1 & \mapsto & a_2 \\ a_2 & \mapsto & a_1 \\ & \vdots & \\ & \vdots & \\ & \vdots & \\ & \vdots & \\ & \vdots & \\ a_{n-1} & \mapsto & a_n \\ a_n & \mapsto & a_{n-1} \end{array} \qquad \begin{array}{rcl} b_1 & \mapsto & b_2 \\ b_2 & \mapsto & b_3 \\ b_3 & \mapsto & b_1 \\ & \vdots & \\ & \vdots & \\ & \vdots & \\ b_{n-2} & \mapsto & b_{n-1} \\ b_{n-1} & \mapsto & b_n \\ b_n & \mapsto & b_{n-2} \end{array}$$

Each map listed above can be viewed as a directed edge. Our Bi-Eulerian path p on D is given by first reading off the edge $e_1 = (a_1a_2)$ in the left-hand column of maps above. This edge has initial vertex a_1 and terminal vertex a_2. We find the b_i that is equal to a_2 and read off the map equivalent to the edge e_2 whose initial vertex is b_i in the right-hand column of maps above. So far our path is e_1e_2. The rest of the path is found by repeating these steps until we are forced to return to the map we started with, at which point we stop.

Clearly p is a Bi-Eulerian path. It must traverse every edge exactly once as it is deduced from the permutations A and B, which were defined by all edges in the graph. It must start and end at the same point as the way we define p follows the method of composing A and B, and we know that $C = AB$ is transitive.

Let $\Gamma = (V, E)$ be a cubic graph of order $2k$ with cyclic orientation at the vertices. Any v_i in V has valency 3. Associate the labels $3(i-1)+1$, $3(i-1)+2$ and $3(i-1)+3$ with each v_i. Choose a unique label from this set to assign to the end of each e_j in E that is incident to v_i. Then, every e_j in E has a unique label at its initial and terminal vertices.

If e_j has label a_k at its initial vertex and a_l at its terminal vertex, define a 2-tuple $(a_k a_l)$. Do this for all edges in E and we have the following list of 2-tuples. There are $3k$ of them.

$$(a_1 a_2), (a_3 a_4), \ldots, (a_{6k-1} a_{6k})$$

Define $A = (a_1 a_2)(a_3 a_4) \ldots (a_{6k-1} a_{6k})$.

Suppose that, at the vertex v_i, the orientation sends the end of edge e_{j_1} to the end of edge e_{j_2}; end of edge e_{j_2} to the end of edge e_{j_3}; end of edge e_{j_3} to the end of edge e_{j_1}. If the labels of the ends of the edges e_{j_1}, e_{j_2} and e_{j_3} that are incident to v_i are b_1, b_2 and b_3, respectively; then define a 3-tuple $(b_1 b_2 b_3)$. Do this for all vertices in V and we have the following list of 3-tuples. There are $2k$ of them.

$$(b_1 b_2 b_3), (b_4 b_5 b_6), \ldots, (b_{6k-2} b_{6k-1} b_{6k})$$

Define $B = (b_1 b_2 b_3)(b_4 b_5 b_6) \ldots (b_{6k-2} b_{6k-1} b_{6k})$.

The permutation $C = AB$ is transitive as composing A and B is an identical process to the one followed in Algorithm 4.3. We know that a Bi-Eulerian path traverses every edge exactly once and starts and ends at the same point. Hence, C is transitive. We blow up Γ by expanding vertices according to the maps defined in Fig. 26, where the white vertex has anti-clockwise cyclic orientation and the black vertex has clockwise cyclic orientation.

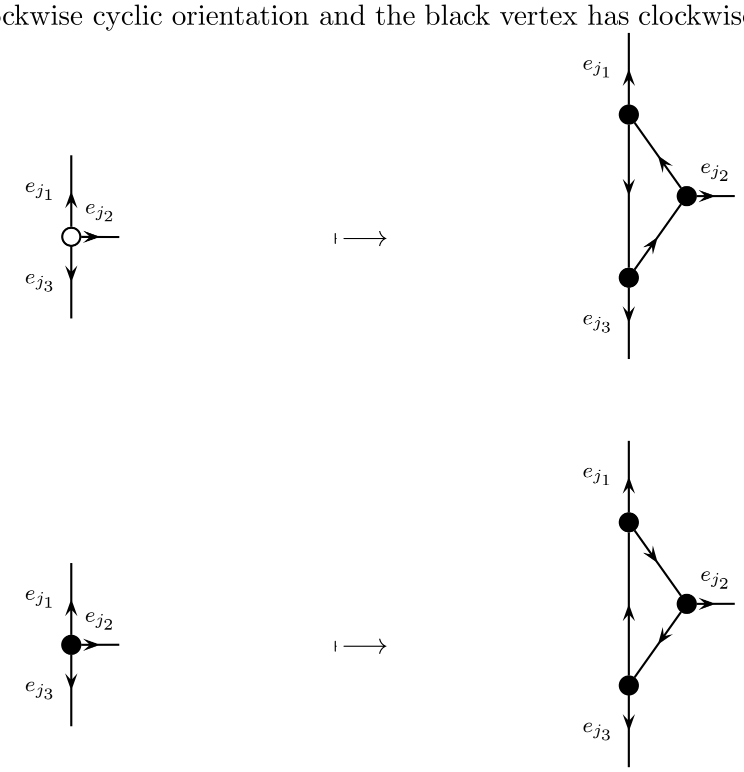


Fig. 26

We then expand each e_{j_i} to a double edge, endowing the new edges with the opposite orientations. For instance, if $e_{j_i} = (x, y)$ then the double of e_{j_i} is $e^d_{j_i} = (y, x)$. We colour all double edges red and all single edges green and we are left with an n-configuration. □

Theorem 4.3. *There is a one-to-one correspondence between conjugacy classes of free subgroups of index $n = 12g - 6$ in M, with related finite transitive triple (Ω, A, B), and MOWFs of genus g.*

Proof. Any subgroup of M has a coset diagram. All coset diagram of subgroups of order n in M can be found by finding all n-configurations and placing them into equivalence classes $\mathcal{E}_n$, with respect to the equivalence relation $\sim_P$. We can then find a representative coset diagram for all subgroups in a given conjugacy class by putting the diagrams in $\mathcal{E}_n$ into equivalence classes with respect to the equivalence relation $\sim_I$. We know that the conjugacy classes of free subgroups of index n will be described by those equivalence classes of non-degenerate n-configurations. Call the set of all such equivalence classes $\mathcal{F}_n$.

Use Algorithm 4.1 to find permutations A and B such that $A^2 = B^3 = 1$ for each F in $\mathcal{F}_n$. Disregard all diagrams where $C = AB$ is not transitive. Then, we are left with the set of equivalence class of n-configurations that represent all conjugacy classes of free subgroups of index n in M that have related finite transitive triple (Ω, A, B), where $\Omega = \{1, 2, \ldots, n\}$.

We can find a cubic graph of order $2k$ with cyclic orientation at the vertices for each of the equivalence classes we have left by applying Algorithm 4.2 to a representative n-configuration for each equivalence class. We then apply Algorithm 4.3 to all cubic graph of order $2k$ we find to obtain a MOWF of genus g associated with each conjugacy class of free subgroups of index $n = 12g - 6$ in M with associated finite transitive triple.

Let w be an MOWF of length $n = 12g - 6$, where g is the genus of the Wicks form. Then, using Algorithm 4.4 we can find a cubic graph of order $2k$ with cyclic orientation at the vertices. Lemma 4.1 tells us that all cubic graphs of order $2k$ with cyclic orientation at the vertices are in bijection with equivalence classes n-configurations that have an associated finite transitive triple. Hence, MOWFs of length $n = 12g - 6$ are in bijection with conjugacy classes of free subgroups of index n in M that have an associated finite transitive triple. □

Hence, as the number of MOWFs of genus g is the quantity M_1^g from Theorem 2.1 then we know that the number of conjugacy classes of free subgroups of index $n = 12g - 6$ in M that have associated transitive triple is also M_1^g. We give M_1^g for the first 15 values of g in the table below.

Table: The number of free subgroups of index $n = 12g - 6$ in M that have an associated transitive triple for $g = 1, \ldots, 15$.

g	M_g^1
1	1
2	9
3	1726
4	1349005
5	2169056374
6	5849686966988
7	23808202021448662
8	136415042681045401661
9	1047212810636411989605202
10	10378926166167927379808819918
11	129040245485216017874985276329588
12	1966895941808403901421322270340417352
13	36072568973390464496963227953956789552404
14	783676560946907841153290887110277871996495020
15	19903817294929565349602352185144632327980494486370

Bibliography

BACHER, ROLAND & VDOVINA, ALINA 2002 Counting 1-vertex triangulations of oriented surfaces. *Discrete Math.* **246** (1-3), 13–27, formal power series and algebraic combinatorics (Barcelona, 1999).

BRENNER, J. L. & LYNDON, R. C. 1983 Permutations and cubic graphs. *Pacific J. Math.* **104** (2), 285–315.

COMERFORD, JR., LEO P. & EDMUNDS, CHARLES C. 1994 Products of commutators and products of squares in a free group. *Internat. J. Algebra Comput.* **4** (3), 469–480.

COUETTE, M. F. A. 1890 Études sur le frottement des liquides. *Annales de Chimie et de Physique* **21**, 433–510.

CULLER, MARC 1981 Using surfaces to solve equations in free groups. *Topology* **20** (2), 133–145.

DIXON, JOHN D. & MORTIMER, BRIAN 1996 *Permutation groups*, *Graduate Texts in Mathematics*, vol. 163. New York: Springer-Verlag.

MILLINGTON, M. H. 1969 Subgroups of the classical modular group. *J. London Math. Soc. (2)* **1**, 351–357.

MUSHTAQ, Q. 1983 Coset diagrams for the modular group. Thesis.

SERRE, J.-P. 1973 *A course in arithmetic*. New York: Springer-Verlag, translated from the French, Graduate Texts in Mathematics, No. 7.

STOTHERS, W. W. 1974 Subgroups of the modular group. *Proc. Cambridge Philos. Soc.* **75**, 139–153.

STOTHERS, W. W. 1977/78 The number of subgroups of given index in the modular group. *Proc. Roy. Soc. Edinburgh Sect. A* **78** (1-2), 105–112.

STOTHERS, W. W. 1978 Free subgroups of the free product of cyclic groups. *Math. Comp.* **32** (144), 1274–1280.